Derivatives

Binomial Options Pricing Model

cesariomateus@gmail.com

www.cesariomateus.com

Binomial Trees

Useful and very popular technique for pricing an option.

Diagram representing different possible paths that might be followed by the stock price over the life of an option.

Underlying assumption: stock price follows a *random walk*

In each time step:
Certain probability of moving up or down by a percentage amount

In the limit, as the time step becomes smaller, this model leads to the lognormal assumption for stock prices that underlies the Black-Scholes model.

Binomial tress can be used to value options using both no-arbitrage arguments and the risk-neutral valuation principle.

Example

Initial stock price = \$20 and it is known that at the end of 3 months will be either: \$22 or \$18.

What is the value of an European call option to buy the stock for \$21 in 3 months?

Stock price = \$20

Stock Price = \$22

Option price = \$1

Stock Price = \$18

Option price = \$0

Arbitrage opportunities do not exist (assumption)

No uncertainty about the value of the portfolio at the end of the 3 months

Portfolio has no risk, the return must be equal the risk-free interest rate

Setting up a Riskless Portfolio

Long position in Δ shares of stock and a short position in one call option

What is the value of Δ that makes the portfolio riskless?

Sock prices moves up from \$20 to \$22, the value of the shares is 22 Δ
The value of the option is 1. Total value of portfolio is 22 Δ -1.

Sock prices moves up from \$20 to \$18, the value of the shares is 18 Δ
The value of the option is 0. Total value of portfolio is 18 Δ -0.

Portfolio is riskless, if the value of Δ is chosen so that the final value of the portfolio is the same (for both alternatives)

$$22\Delta - 1 = 18\Delta \Leftrightarrow \Delta = 0.25$$

Riskless portfolio: long 0.25 shares, short 1 option.

$$\text{Portfolio Value} = 22 \times 0.25 - 1 = 4.5 \text{ or } 18 \times 0.25 = 4.5$$

Riskless portfolios must, in absence of arbitrage opportunities earn the risk-free rate of interests.

Suppose risk-free rate in 12% per annum. The value of the portfolio today is the present value of 4.5:

$$4.5^{-0.12\times3/12} = 4.367$$

The value of the option today (*f*):

$$20 \times 0.25 - f = 4.367 \Leftrightarrow f = 0.633$$

If the value of the option is more than 0.633, the portfolio would cost less than 4.367 to set up and would earn more than the risk-free rate

If the value of the option is less than 0.633, shorting the portfolio would provide a way of borrowing money at less than the risk free rate.

Generalization

A derivative lasts for time T and is dependent on a stock

S_0= Stock Price
f= option price

$$\begin{tikzcd} & \begin{array}{c} S_0u \\ f_u \end{array} \\ \begin{array}{c} S_0 \\ f \end{array} \arrow[ur] \arrow[dr] & \\ & \begin{array}{c} S_0d \\ f_d \end{array} \end{tikzcd}$$

Portfolio: Long position in Δ shares and a short position in one option

Value of the portfolio at the end of life of the option if there is an up movement in the stock price

$$S_0u\Delta - f_u$$

Value of the portfolio at the end of life of the option if there is a down movement in the stock price

$$S_0d\Delta - f_d$$

The two are equal when:

$$S_0 u\Delta - f_u = S_0 d\Delta - f_d$$
$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

Portfolio is riskless, no arbitrage opportunities, it must earn the risk-free interest rate.
Present value of the portfolio:

$$(S_0 u\Delta - f_u)e^{-rT}$$

The cost of setting the portfolio:

$$S_0\Delta - f$$

It follows that:

$$S_0\Delta - f = (S_0 u\Delta - f_u)e^{-rT} \; or \; f = S_0\Delta(1 - ue^{-rT}) + f_u e^{-rT}$$

The equation can be reduced to:

$$f = e^{-rT}[pf_u + (1-p)f_d]$$

$$p = \frac{e^{rT} - d}{u - d}$$

Option Price (one-step binomial tree):

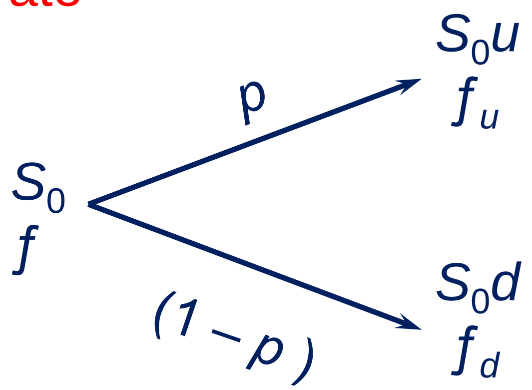
$$p = \frac{e^{0.12\times 3/12} - 0.9}{1.1 - 0.9} = 0.6523$$

$$f = e^{-0.12\times 0.25}(0.6523 \times 1 + 0.3477 \times 0) = 0.633$$

p as a Probability

It is natural to interpret p and $1\text{-}p$ as probabilities of up and down movements

The value of a derivative is then its expected payoff in a risk-neutral world discounted at the risk-free rate



Risk-Neutral Valuation

When the probability of an up and down movements are p and $1-p$ the expected stock price at time T is S_0e^{rT}

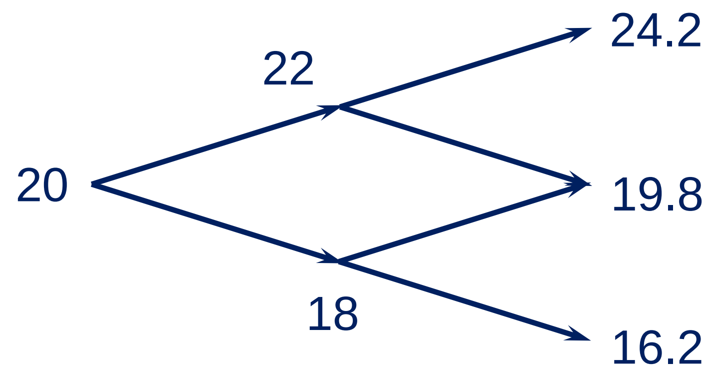
This shows that the stock price earns the risk-free rate

Binomial trees illustrate the general result that to value a derivative we can assume that the expected return on the underlying asset is the risk-free rate and discount at the risk-free rate

This is known as using risk-neutral valuation

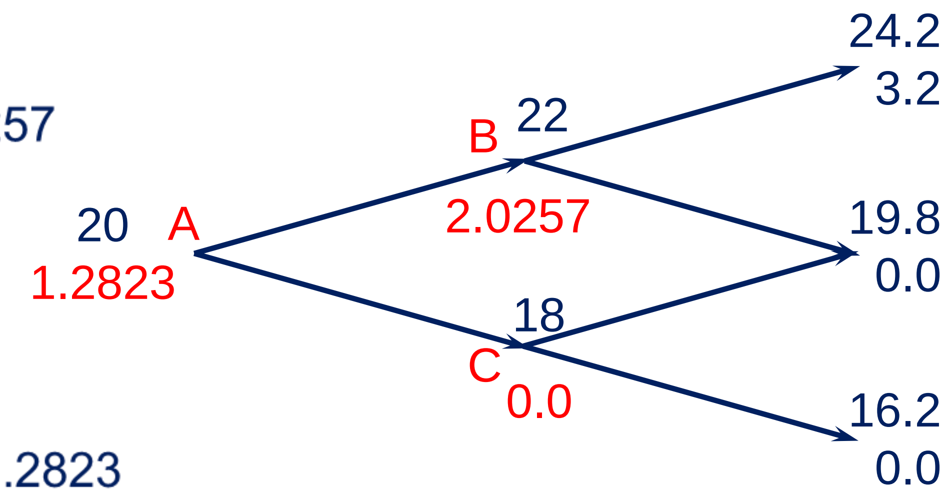
Two-Step Binomial Trees

$u = d = 10\%$
X = 21
R = 12%
Each time step is 3 months



Value at node B

$$e^{-0.12\times0.25}(0.6523\times3.2 + 0.3477\times0) = 2.0257$$

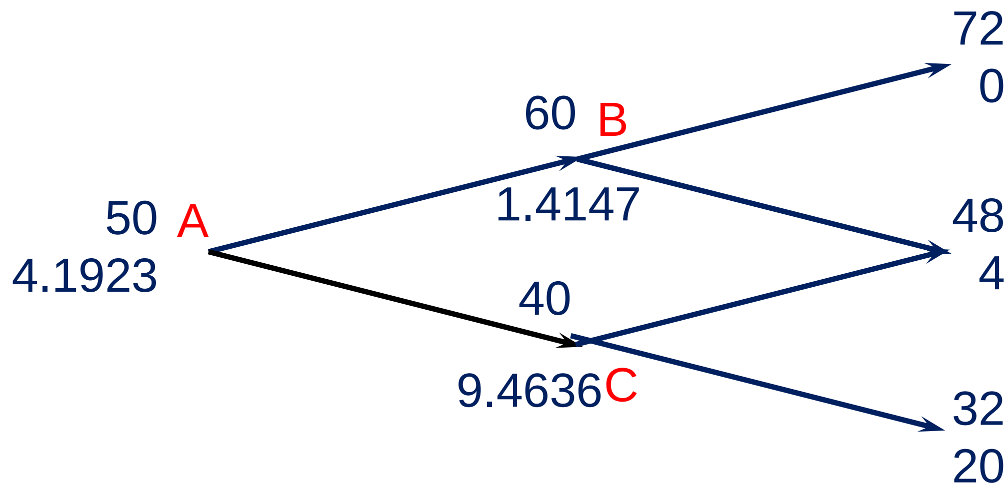


Value at node A

$$e^{-0.12\times0.25}(0.6523\times2.0257 + 0.3477\times0) = 1.2823$$

Example with Put

$S_0 = 50$
$K = 52$
r = 5%
$u = 1.20$
$d = 0.8$
$p = 0.6282$
Each time step is one year



Value at node B

$e^{-0.05\times1}(0.6282\times0 + 0.3718\times4) = 1.41467$

Value at node C

$e^{-0.05\times1}(0.6282\times4 + 0.3718\times20) = 9.46359$

Value at node A

$e^{-0.05\times1}(0.6282\times1.4147 + 0.3718\times9.4636) = 4.19233$

American Options

The value of the option at the final nodes is the same as for European Options.

At earlier nodes the value of the option is the greater of:
1. The value given by:

$$f = e^{-r\Delta t}[pf_u + (1-p)f_d]$$

2. The payoff from early exercise

$$\mathit{Value\ at\ node\ B}$$

$e^{-0.05\times1}(0.6282\times0 + 0.3718\times4) =$ 1.41467 since early exercice value is (-8)

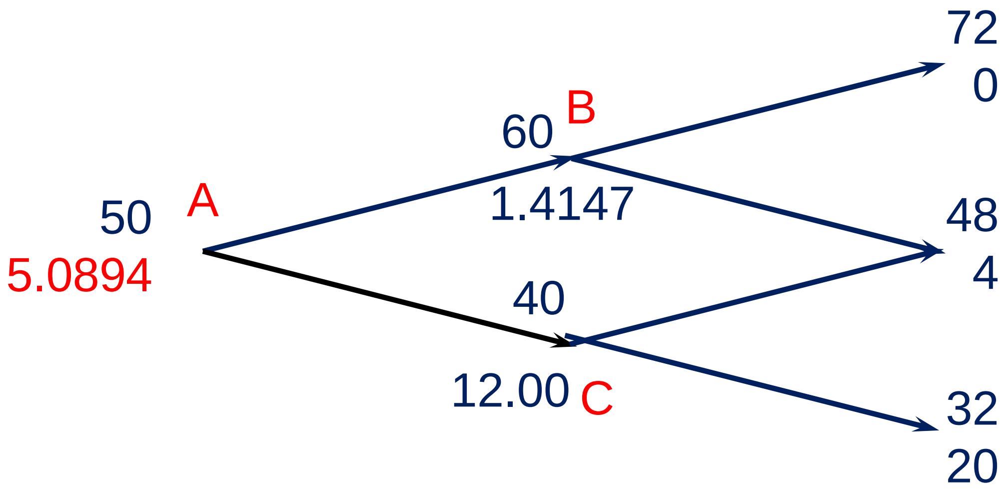
$$\mathit{Value\ at\ node\ C}$$

Payoff early exercise is 12

$e^{-0.05\times1}(0.6282\times4 + 0.3718\times20) =$ 9.46359

$$\mathit{Value\ at\ node\ A}$$

$e^{-0.05\times1}(0.6282\times1.4147 + 0.3718\times12.00) =$ 5.0894



At the initial node A, the value is 5.0894 and the payoff from early exercise is 2

Early exercise is not optimal.