Chapter 5

1. Find the first-order partial derivatives $\partial y/\partial x$, $\partial y/\partial z$, $\partial y/\partial t$ of:

$$y = 11x^2zt - 5z^3t^2 + 9xzt^3 + 3x^2z^2$$

$$\frac{\partial y}{\partial x} = 22xzt + 9zt^3 + 6xz^2$$

$$\frac{\partial y}{\partial z} = 11x^2t - 15z^2t^2 + 9xt^3 + 6x^2z$$

$$\frac{\partial y}{\partial t} = 11x^2z - 10z^3t + 27xzt^2$$

2. Use appropriate rules to find the first-order partials for each of the following:

(i) $z = (w - x - y)(3w + 2x - 4y)$

$$\frac{\partial z}{\partial w} = (w - x - y)(3) + (3w + 2x - 4y)(1) = \mathbf{6w - x - 7y}$$

$$\frac{\partial z}{\partial x} = (w - x - y)(2) + (3w + 2x - 4y)(-1) = \mathbf{-w - 4x + 2y}$$

$$\frac{\partial z}{\partial y} = (w - x - y)(-4) + (3w + 2x - 4y)(-1) = \mathbf{-7w + 2x + 8y}$$

(ii) $z = \frac{x^2-y^2}{3x+2y}$

$$\frac{\partial z}{\partial x} = \frac{2x(3x + 2y) - (x^2 - y^2)(3)}{(3x + 2y)^2} = \frac{\mathbf{3x^2 + 4xy + 3y^2}}{\mathbf{(3x + 2y)^2}}$$

$$\frac{\partial z}{\partial y} = \frac{-2y(3x + 2y) - (x^2 - y^2)(2)}{(3x + 2y)^2} = \frac{\mathbf{-2x^2 = -6xy - 2y^2}}{\mathbf{(3x + 2y)^2}}$$

(iii) $z = (5x^2 - 4y)^2(2x + 7y^3)$

Using the product rule and generalized power function rule:

$$\frac{\partial z}{\partial x} = (5x^2 - 4y)^2(2) + (2x + 7y^3)[2(5x^2 - 4y)(10x)]$$
$$= \mathbf{2(5x^2 - 4y^2) + (2x + 7y^3)[100x^3 - 80xy]}$$

$$\frac{\partial z}{\partial y} = (5x^2 - 4y)^2(21y^2) + (2x + 7y^3)[2(5x^2 - 4y)(-4)]$$
$$= \mathbf{(21y^2)(5x^2 - 4y)^2 + (2x + 7y^3)(-40x^2 + 32y)}$$

3. Take the first, second, and cross partial derivatives of the following functions:

(i) $z = 7x^3 + 9xy + 2y^5$

$\frac{\partial z}{\partial x} = 21x^2 + 9y$ $\qquad$ $\frac{\partial^2 z}{\partial x^2} = 42x$

$\frac{\partial z}{\partial y} = 9x + 10y^4$ $\qquad$ $\frac{\partial^2 z}{\partial y^2} = 40y^3$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(21x^2 + 9y) = \mathbf{9}$ $\qquad$ $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(9x + 10y^4) = \mathbf{9}$

(ii) $z = e^{3x+2y}$

$\frac{\partial z}{\partial x} = 3e^{3x+2y}$ $\qquad$ $\frac{\partial^2 z}{\partial x^2} = 3e^{3x+2y}(3) = 9e^{3x+2y}$

$\frac{\partial z}{\partial y} = 2e^{3x+2y}$ $\qquad$ $\frac{\partial^2 z}{\partial y^2} = 2e^{3x+2y}(2) = 4e^{3x+2y}$

$\frac{\partial^2 z}{\partial x \partial y} = 6e^{3x+2y} = \frac{\partial^2 z}{\partial y \partial x}$

(iii) $z = \ln(5x + 9y)$

$\frac{\partial z}{\partial x} = \frac{5}{5x+9y}$ $\qquad$ $\frac{\partial z}{\partial y} = \frac{9}{5x+9y}$

By the quotient rule:

$\frac{\partial^2 z}{\partial x^2} = \frac{0(5x+9y)-5(5)}{(5x+9y)^2} = \frac{-25}{(5x+9y)^2}$ $\qquad$ $\frac{\partial^2 z}{\partial y^2} = \frac{0(5x+9y)-9(9)}{(5x+9y)^2} = \frac{-81}{(5x+9y)^2}$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{-45}{(5x+9y)^2} = \frac{\partial^2 z}{\partial y \partial x}$

4. For the following functions**, (a)** find the critical points, and **(b)** determine if at these points the functions is at a relative maximum, relative minimum, inflexion point, or saddle point.

(i) $f(x, y) = 3x^3 + 1.5y^2 - 18xy + 17$

a. Set the first-order partial derivatives equal to zero,

$$f'x = 9x^2 - 18y = 0$$
$$f'y = 3y - 18x = 0$$

and solve the critical values,

$18y = 9x^2 \Leftrightarrow y = 1/2x^2$
$3y = 18x \Leftrightarrow y = 6x$

Setting y equal to $6x$

$\frac{1}{2x^2} = 6x \Leftrightarrow x^2 - 12x = 0 \Leftrightarrow x(x - 12) = 0 \Leftrightarrow \boldsymbol{x = 0}$ or $\boldsymbol{x = 12}$

Setting $x = 0$ and $x = 12$ into $y = 6x$ we get $(0,0)$ and $(12,72)$ as the coordinates of the critical points.

b. Take the second-order partial derivatives, evaluate tem at the critical points and note the signs:

$f''_x = 18x$ $\quad$ $f''yy = 3$
$f''_x(0,0) = 18(0)$ $\quad$ $f''y(0,0) = 3 > 0$
$f''_x(12,72) = 18(12) = 216 > 0$ $\quad$ $f''y(12,72) = 3 > 0$

Then take cross-partials:

$f_{xy} = -18 = fyx$

Evaluate it at the critical point and test the third condition:

$$f_{xx}(a,b) \times f_{yy}(a,b) > [f_{xy}(a,b)]^2$$

At (0.0) $\quad$ $0 \times 3 < (-18)^2$
At (12.72) $\quad$ $216 \times 3 > (-18)^2$

With $f_{xx}f_{yy} > (f_{xy})^2$ and $f_{xx}, f_{yy} > 0$ at $(12,72)$ is a relative minimum.
With $f_{xx}f_{yy} < (f_{xy})^2$ and f_{xx} and f_{xy} of the same sign at $(0,0)$, $f(0,0)$ is an inflexion point.

(ii) $\quad z(x,y) = \ln(x^2 - 4x + 3y^2 - 6y)$

a.

$$z_x = \frac{2x - 4}{x^2 - 4x + 3y^2 - 6y} = 0 \Leftrightarrow 2x - 4 = 0 \Leftrightarrow x = 2$$
$$z_y = \frac{6y - 6}{x^2 - 4x + 3y^2 - 6y} = 0 \Leftrightarrow 6y - 6 = 0 \Leftrightarrow y = 1$$

b.

$$z_{xx} = \frac{(2)(x^2 - 4x + 3y^2 - 6y) - (2x - 4)(2x - 4)}{(x^2 - 4x + 3y^2 - 6y)^2}$$

At $x = 2, y = 1$

$$z_{xx} = \frac{(-7)(2) - 0}{(-7)^2} = \frac{-14}{49} < 0$$
$$z_{yy} = \frac{(-7)(6) - 0}{(-7)^2} = \frac{-42}{-49} < 0$$

Then,

$$z_{xy} = \frac{-(2x - 4)(6y - 6)}{(x^2 - 4x + 3y^2 - 6y)^2} = z_{yx}$$

At $x = 2, y = 1, z_{xy} = 0 = z_{yx}$. With z_{xx}, z_{yy}<0, and $z_{xx} \times z_{yy} > (z_{yx})^2$ the function is at a **Maximum**

5. Find the total differential dz for the function $z = (x - 3y)^3$

$z_x = 3(x - 3y)^2(1)$ $\qquad$ $z_y = 3(x - 3y)^2(-3)$

$$dz = 3(x - 3y)^2 dx - 9(x - 3y)^2 dy$$

6. In monopolistic competition producers must determine the price that will maximize their profit. Assume that a producer offers two different brands of a product, for which the demand functions are:

$$Q_1 = 14 - 0.25P_1$$
$$Q_2 = 24 - 0.5P_2$$

And the joint cost function is:

$$TC = Q_1^2 + 5Q_1Q_2 + Q_2^2$$

Find:

(i) The profit-maximizing level of output for each product
(ii) The profit-maximizing price for each product
(iii) The maximum profit
(Hint: Total Revenue for the firm is $P_1Q_1 + P_2Q_2$)

(i) $M = P_1Q_1 + P_2Q_2 - TC \Leftrightarrow M = P_1Q_1 + P_2Q_2 - (Q_1^2 + 5Q_1Q_2 + Q_2^2)$

From
$Q_1 = 14 - 0.25P_1 \Rightarrow P_1 = 56 - 4Q_1$
$Q_2 = 24 - 0.5P_2 \Rightarrow P_2 = 48 - 2Q_2$

Substituting in the Profit function:
$M = (56 - 4Q_1)Q_1 + (48 - 2Q_2)Q_2 - Q_1^2 - 5Q_1Q_2 - Q_2^2)$
$M = 56Q_1 - 5Q_1^2 + 48Q_2 - 3Q_2^2 - 5Q_1Q_2$

Then, maximize by the familiar rules:
$M_1 = 56 - 10Q_1 - 5Q_2 = 0$
$M_2 = 48 - 6Q_2 - 5Q_1 = 0$

Which them solved simultaneously give
$Q_1 = 2.75$
$Q_2 = 5.7$

Take the second-order derivatives to be sure M is maximized:

$M_{11} = -10$ $\quad M_{22} = -6$ $\quad M_{12} = -5 = M_{21}$

With both second direct partials negative and $M_{11} \times M_{22} > (M_{12})^2$, the function is maximized at the critical values.

(ii) Substitute $Q_1 = 2.75$ and $Q_2 = 5.7$ in $P_1 = 56 - 4Q_1$ and $P_2 = 48 - 2Q_2$, respectively, to find the profit-maximizing price for each product.

$\boldsymbol{P_1} = 56 - 4(2.75) = \mathbf{45}$ for brand 1
$\boldsymbol{P_2} = 48 - 2(5.7) = \mathbf{36.6}$ for brand 2

(ii) Max-profit is:

$\boldsymbol{M} = 45(2.75) + 36.3(5.7) - (2.75)^2 - 5(2.75)(5.7) - (5.7)^2 = \mathbf{213.94}$

7. Find the turning point of the following function and determine whether it is a maximum or minimum.

$y = 3x_1^2 - 5x_1 - x_1x_2 + 6x_2^2 - 4x_2 + 2x_2x_3 + 4x_3^2 + 2x_3 - 3x_1x_3$