Chapter 5

Calculus of Multivariate Functions

FR2200 – Mathematical Finance

Dr Cesario MATEUS

cesario.mateus1@city.ac.uk
www.cesariomateus.com

Functions of several variables and partial derivatives

Many economic and financial activities involve functions of more than one independent variables.

Example:
Production theory is the combination of capital and labour, which determines the output produced, and not just one factor of production:

$$Q = f(K, L)$$

$z = f(x, y)$ is defined as a function of *two independent variables* if there exists one and only one value of z in the range of f for each ordered pair of real numbers (x, y) in the domain of f.

By convention, z is the *dependent variable*; x and y are *independent variables*

Partial Derivative: the measure of the effect of a change in a single independent variable x or y on the dependent variable z in a multivariable function.

The partial derivative of z with respect to x measures the instantaneous rate of change of z with respect to x while y is held constant.

It is written as:

$$\boxed{\partial z/\partial x,\ \partial f/\partial x,\ f_x(x,y),\ f_x \ or\ z_x}$$

Mathematically

$$\boxed{\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,\ y) - f(x,y)}{\Delta x}}$$

Note: Partial differentiation with respect to one of the independent variables follows the same rules as ordinary differentiation, while the other independent variables are treated as constants

Example

Partial derivatives of the multivariable function $z = 3x^2y^3$

$$\frac{\partial z}{\partial x} = 6xy^3 \qquad \frac{\partial z}{\partial y} = 3x^2 3y^2 = 9x^2y^2$$

Partial derivatives of the multivariable function $z = 5x^3 - 3x^2y^2 + 7y^5$

$$\frac{\partial z}{\partial x} = 15x^2 - 6xy^2 \qquad \frac{\partial z}{\partial y} = -3x^2 2y + 35y^4 = -6x^2y + 35y^4$$

Rules of partial Differentiation

Partial derivatives follow the same basic rules as the rules of differentiation. A few key rules are given below:

PRODUCT RULE

Given $z = g(x, y) \times h(x, y)$,

$$\frac{\partial z}{\partial x} = h(x, y) \times \frac{\partial g}{\partial x} + g(x, y) \times \frac{\partial h}{\partial x}$$

$$\frac{\partial z}{\partial y} = h(x, y) \times \frac{\partial g}{\partial y} + g(x, y) \times \frac{\partial h}{\partial y}$$

Example: Given $z = (3x + 5)(2x + 6y)$, then, by the product rule:

$$\frac{\partial z}{\partial x} = (2x + 6y)(3) + (3x + 5)(2) = \boxed{12x + 10 + 18y}$$

$$\frac{\partial z}{\partial y} = (2x + 6y)(0) + (3x + 5)(6) = \boxed{18x + 30}$$

QUOTIENT RULE

Given $z = g(x,y)/h(x,y),\ and\ h(x,y) \neq 0,$

$$\frac{\partial z}{\partial x} = \frac{h(x,y) \times \dfrac{\partial g}{\partial x} - g(x,y) \times \dfrac{\partial h}{\partial x}}{[h(x,y)]^2}$$

$$\frac{\partial z}{\partial y} = \frac{h(x,y) \times \dfrac{\partial g}{\partial y} - g(x,y) \times \dfrac{\partial h}{\partial y}}{[h(x,y)]^2}$$

Example: Given $z = (6x + 7y)/(5x + 3y)$, by the quotient rule,

$$\frac{\partial z}{\partial x} = \frac{6(5x + 3y) \; - \; (6x + 7y)(5)}{(5x + 3y)^2}$$

$$= \frac{30x + 18y - 30x - 35y}{(5x + 3y)^2} = \frac{-17y}{(5x + 3y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{7(5x + 3y) \; - \; (6x + 7y)(3)}{(5x + 3y)^2}$$

$$= \frac{35x + 21y - 18x - 21y}{(5x + 3y)^2} = \frac{17x}{(5x + 3y)^2}$$

GENERALIZED POWER FUNCTION RULE

Given $z = [g(x, y)]^n$,

$$\frac{\partial z}{\partial x} = n[g(x, y)]^{n-1} \times \frac{\partial g}{\partial x}$$

$$\frac{\partial z}{\partial y} = n[g(x, y)]^{n-1} \times \frac{\partial g}{\partial y}$$

Example: Given $z = (x^3 + 7y^2)^4$, by the generalized power function rule,

$$\boxed{\frac{\partial z}{\partial x} = 4(x^3 + 7y^2)^3 \times (3x^2) = 12x^2(x^3 + 7y^2)^3}$$

$$\boxed{\frac{\partial z}{\partial y} = 4(x^3 + 7y^2)^3 \times (14y) = 56y(x^3 + 7y^2)^3}$$

Second-order partial derivatives

Given a function $z = f(x, y)$, the second-order direct partial derivative signifies that the function has been differentiated partially with respect to one of the independent variables twice while the other independent variable has been held constant:

$$\boxed{f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}}$$

$$\boxed{f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}}$$

f_{xx} measures the rate of change of the first-order partial derivative f_x with respect to x while y is held constant.

Cross (or mixed) partial derivatives

f_{xy} and f_{yx} indicate that first the primitive function has been partially differentiated with respect to one independent variable and then that partial derivative has in turn been partially differentiated with respect to the other independent variable.

$$\boxed{f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}}$$

$$\boxed{f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}}$$

Cross partial: measures the rate of change of a first-order partial derivative with respect to the other independent variable.

Example

Find the first, second, and cross-partial derivatives of the function:

$$z = 3x^2y^3$$

And then evaluate them at $x = 4; y = 1$

$$z_x = 6xy^3$$
$$z_x(4,1) = 6(4)(1)^3 = 24$$

$$z_y = 9x^2y^2$$
$$z_y(4,1) = 9(4)^2(1)^2 = 144$$

$$z_{xx} = 6y^3$$
$$z_{xx}(4,1) = 6(1)^3 = 6$$

$$z_{yy} = 18x^2y$$
$$z_{yy}(4,1) = 18(4)^2(1) = 288$$

$$z_{xy} = \frac{\partial}{\partial y}(6xy^3) = 18xy^2$$
$$z_{xy}(4,1) = 18(4)(1)^2 = 72$$

$$z_{yx} = \frac{\partial}{\partial x}(9x^2y^2) = 18xy^2$$
$$z_{yx}(4,1) = 18(4)(1)^2 = 72$$

Note that the two cross partial derivatives are identical. This results is known as Young’s Theorem

Example:

Find all the first and second partial derivatives of the function $z = e^{x^2+y^2}$

The first partial derivatives are:

$$z_x = 2xe^{x^2+y^2}$$

$$z_y = 2ye^{x^2+y^2}$$

To partially differentiate the expressions above, we need to employ the product rule:

$$\text{Let} \quad u = 2x \;, \quad v = e^{x^2+y^2}$$

$$u = 2y \;, \quad v = e^{x^2+y^2}$$

$$z_{xx} = 2(e^{x^2+y^2}) + 2x(2xe^{x^2+y^2}) = 2e^{x^2+y^2}(2x^2+1)$$

$$z_{yy} = 2(e^{x^2+y^2}) + 2y(2ye^{x^2+y^2}) = 2e^{x^2+y^2}(2y^2+1)$$

The cross-partial derivatives are:

$$\boxed{z_{xy} = 4xye^{x^2 + y^2} = z_{yx}}$$

Notice that for the cross-partial derivatives the product rule is not required

Optimization of Multivariate Functions

Three conditions should be met for a function such as:

$$y = f(x, z)$$

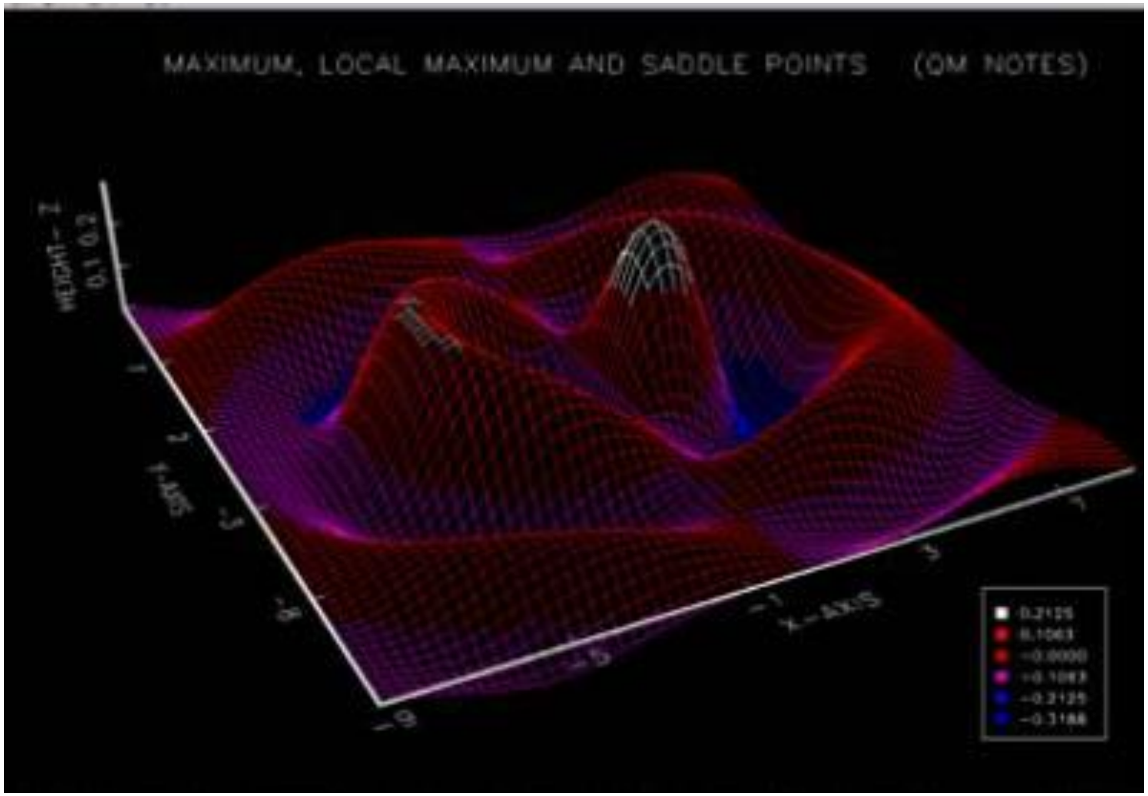
To be at a relative maximum, or minimum

1) First-order partial derivatives must equal zero simultaneously. In other words, this is the *first-order* (necessary) condition for a stationary value (at a critical point (a, b) the function is neither increasing nor decreasing

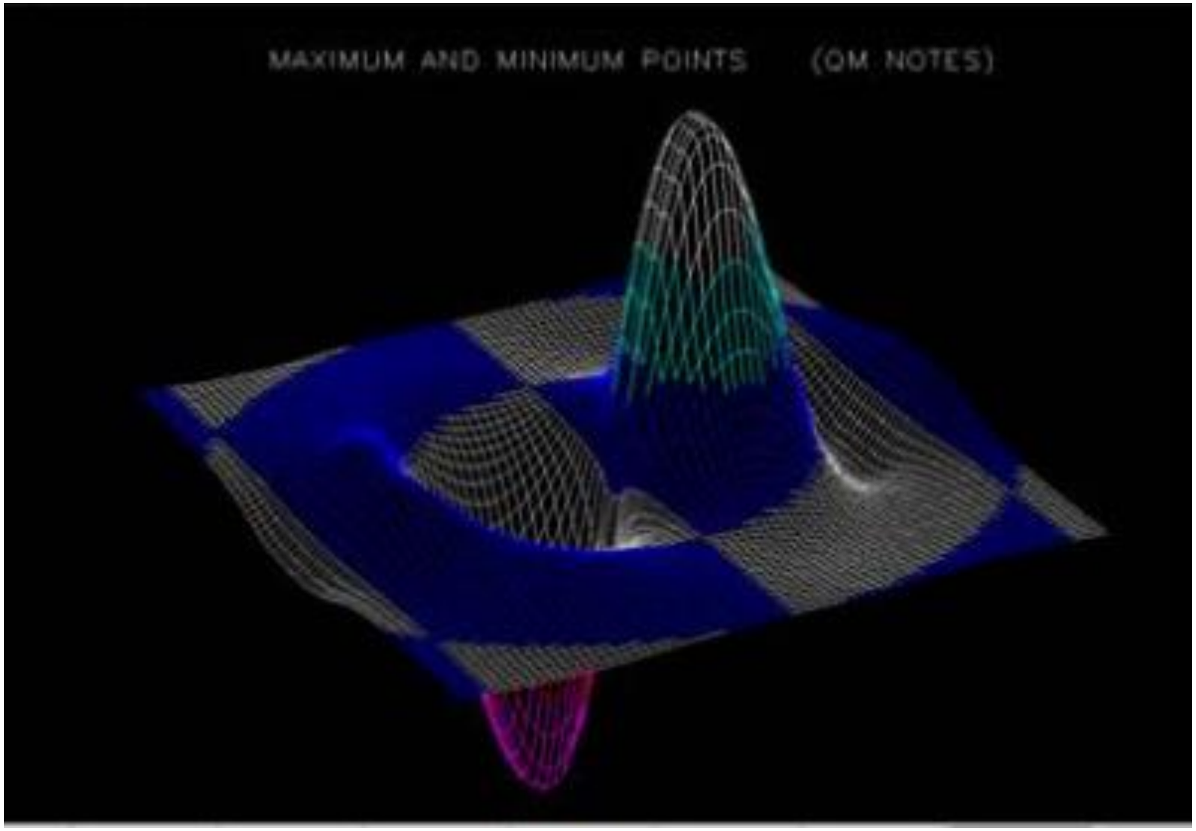
2) The second-order direct partial derivatives (when evaluate at the critical point (a, b) must both be *positive for a minimum* and *negative for a maximum*. This ensures that from a relative plateau at (a, b) the function is *moving upward* in relation to the principal axes in *the case of a minimum*, and *downward* in relation to the principal axes in the case of a *maximum*.

3) The product of the second-order direct partials evaluated at the critical point must exceed the product of the cross partials evaluated at the critical point.

Points 2 and 3 before together describe the second-order (sufficient) conditions for a relative extremum.



Plot of a Multivariate Function



Maximum and Minimum

TURNING POINT TO BE	First Order Necessary Condition	*Second Order Sufficient Condition	
Maximum	$\frac{\partial y}{\partial x} = 0$ and $\frac{\partial y}{\partial z} = 0$	$\frac{\partial^2 y}{\partial x^2} < 0$ and $\frac{\partial^2 y}{\partial z^2} < 0$	$\frac{\partial^2 y}{\partial x^2} \cdot \frac{\partial^2 y}{\partial z^2} > \left(\frac{\partial^2 y}{\partial z \partial x} \right)^2$
Minimum	$\frac{\partial y}{\partial x} = 0$ and $\frac{\partial y}{\partial z} = 0$	$\frac{\partial^2 y}{\partial x^2} > 0$ and $\frac{\partial^2 y}{\partial z^2} > 0$	$\frac{\partial^2 y}{\partial x^2} \cdot \frac{\partial^2 y}{\partial z^2} > \left(\frac{\partial^2 y}{\partial z \partial x} \right)^2$
Saddle Point	$\frac{\partial y}{\partial x} = 0$ and $\frac{\partial y}{\partial z} = 0$	$\frac{\partial^2 y}{\partial x^2} > 0$, $\frac{\partial^2 y}{\partial z^2} < 0$ **or vice versa**	

*** Applies only if the first order necessary condition is met**

Note the following

(a) Since $f_{xy} = f_{yx}$ by Young's theorem, $f_{xy} * f_{yx} = (f_{xy})^2$ (see step 3)

(b) If $f_{xx} * f_{yy} \prec (f_{xy})^2$, when f_{xx} and f_{yy} have the same signs, the function is at an *inflection point*. When f_{xx} and f_{yy} have different signs, the function is at a *saddle point*, i.e. the function is at a maximum when viewed from one axis but a minimum when viewed from the other axis.

(c) If $f_{xx} * f_{yy} = (f_{xy})^2$ the test is inconclusive.

Example

Find the critical points of the function

$$z = 2y^3 - x^3 + 147x - 54y + 12$$

Test whether the function is at a relative maximum or minimum

i) Take the first-order partial derivatives, set them equal to zero, and solve for y and x

$$\boxed{\begin{aligned} z_x &= -3x^2 + 147 = 0 \\ x^2 &= 49 \\ x &= \pm 7 \end{aligned}} \qquad \boxed{\begin{aligned} z_y &= 6y^2 - 54 = 0 \\ y^2 &= 9 \\ y &= \pm 3 \end{aligned}}$$

With $x = \pm 7$ and $y = \pm 3$, there are four distinct sets of critical points: $\boxed{(7,3),\ (7,-3),\ (-7,\ 3),(-7,-3)}$.

ii) Take the second-order direct partials, evaluate them at each of the critical points, and check their signs

$$\boxed{z_{xx} = -6x} \qquad \boxed{z_{yy} = 12y}$$

(1)	$z_{xx}(7, 3) = -6(7) = -42 \prec 0$	$z_{yy}(7, 3) = 12(3) = 36 \succ 0$
(2)	$z_{xx}(7, -3) = -6(7) = -42 \prec 0$	$z_{yy}(7, -3) = 12(-3) = -36 \prec 0$
(3)	$z_{xx}(-7, 3) = -6(-7) = 42 \succ 0$	$z_{yy}(-7, 3) = 12(3) = 36 \succ 0$
(4)	$z_{xx}(-7, -3) = -6(-7) = 42 \succ 0$	$z_{yy}(-7, -3) = 12(-3) = -36 \prec 0$

1 and 4: function cannot be relative maximum or minimum (different signs for each of direct partials).

when z_{xx} and z_{yy} are of different signs, $z_{xx} \times z_{yy}$ cannot be greater than $(z_{xy})^2$, and the function is at a saddle point.

2 and 3: function can be relative maximum (7, -3) or minimum (-7, 3) (different signs for each of direct partials).

However, third condition must be tested first to ensure against the possibility of an inflection point.

Need to take cross partial derivatives and make sure that:

The function is maximized at (7, -3) and minimized at (-7, 3)

$$z_{xy} = 0 \qquad z_{yx} = 0$$

From (2), $(-42)(-36) = 1512 \succ (0)^2$

From (3), $(42)(36) = 1512 \succ (0)^2$

The function is maximized at (7,-3) and minimized at (-7, 3)

Optimization of Multivariate Functions in Economics

Example: A firm producing two goods x and y has the following profit function:

$$\pi = 64x - 2x^2 - 4xy - 4y^2 + 32y - 14$$

i) Take the first-order partial derivatives, set them equal to zero, and solve for x and y

$$\boxed{\begin{aligned} \pi_x &= 64 - 4x + 4y = 0 \\ \pi_y &= 4x - 8y + 32 = 0 \end{aligned}}$$

When solved simultaneously: $\boxed{\bar{x} = 40 \text{ and } \bar{y} = 24}$

ii) Take the second-order partial derivatives since both must be negative for the function to be maximum

$$\boxed{\pi_{xx} = -4 \qquad\qquad \pi_{yy} = -8}$$

iii) Take the cross-partials to make sure

$$\pi_{xx}\pi_{yy} > \left(\pi_{xy}\right)^2$$

$$\pi_{xy} = \pi_{yx} = 4$$

$$\pi_{xx}\pi_{yy} > \left(\pi_{xy}\right)^2$$

$$(-4)(-8) > (4)^2$$
$$32 > 16$$

Profit are indeed maximized at $\bar{x} = 40$ and $\bar{y} = 40$. At that point, $\bar{\pi} = 1650$

Critical Points of Multivariate Functions: The General Case

Consider a function $y = f(x) = f(x_1, x_2, \dots x_n)$.
We define the gradient vector (or simply gradient) as:

$$\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \begin{pmatrix} \partial y / \partial x_1 \\ \partial y / \partial x_2 \\ \vdots \\ \partial y / \partial x_N \end{pmatrix}$$

We define the Hessian as the matrix containing the second partial derivatives and cross partial derivatives; it is a square symmetric matrix with the following form

$$H = \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x} \partial \mathbf{x}'} = \begin{pmatrix} \partial^2 y / \partial x_1 \partial x_1 & \partial^2 y / \partial x_1 \partial x_2 & \cdots & \partial^2 y / \partial x_1 \partial x_N \\ \partial^2 y / \partial x_2 \partial x_1 & \partial^2 y / \partial x_2 \partial x_2 & \cdots & \partial^2 y / \partial x_2 \partial x_N \\ \vdots & & \ddots & \vdots \\ \partial^2 y / \partial x_N \partial x_1 & \partial^2 y / \partial x_N \partial x_2 & \cdots & \partial^2 y / \partial x_N \partial x_N \end{pmatrix}$$

The necessary and sufficient conditions for the maximization (minimisation) problem in the multivariate case are a generalization of the two variable case.

First Order Condition (FOC)

For a general function of n variables, say $y = f(x_1, \dots, x_n)$, the FOC for a turning point amounts to:

$$\boxed{\frac{\partial y}{\partial x_1} = \frac{\partial y}{\partial x_2} = \dots = \frac{\partial y}{\partial x_n} = 0}$$

Second Order Condition (FOC)
Let the Hessian be:

$$\boxed{H = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix}}$$

Where:

$$\boxed{f_{11} = \frac{\partial^2 y}{\partial x_1^2},\ f_{12} = \frac{\partial^2 y}{\partial x_1 \partial x_2},\ f_{22} = \frac{\partial^2 y}{\partial x_2^2},\ f_{21} = \frac{\partial^2 y}{\partial x_2 \partial x_1}, \quad \text{etc}}$$

Denote successive principal minors of **H** by:

$$|H_1| = f_{11},\quad |H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix},\ \ldots,\ |H_n| = |H|$$

Then, the SOCs are as follows:

for a maximum, H must be negative definite. Equivalently:

$$|H_1| < 0,\quad |H_2| > 0,\quad |H_3| < 0,\ \ldots$$

for a minimum H must be positive definite; that is:

$$|H_1| > 0, |H_2| > 0,\quad |H_3| > 0,\ \ldots$$

Example

Let: $\boxed{z = 3x^2 - xy + 2y^2 - 4x - 7y + 12}$

$$z_x = 6x - y - 4 = 0$$
$$z_y = -x + 4y - 7 = 0$$

Solving the above two direct partials simultaneously we get the critical point at which the function is optimized $(x = 1,\ y = 2)$.

The second-order partials are $z_{xx} = 6,\ z_{yy} = 4,\ z_{xy} = -1$. Using the Hessian to test the second-order conditions,

$$|H_1| = \begin{vmatrix} z_{xx} & z_{yy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 6 & -1 \\ -1 & 4 \end{vmatrix}$$

Taking the principal minors, $|H_1| = 6 > 0$

and $|H_2| = \begin{vmatrix} 6 & -1 \\ -1 & 4 \end{vmatrix} = 6(4) - (-1)(-1) = 23 > 0$

With $|H_1| > 0$ and $|H_2| > 0$, the Hessian $|H|$ is positive definite, and z is minimized at the critical values.

Example

Consider the function: $y = x_1^2 - 2x_1x_2 + 2x_2^2 + 2x_1x_3 + 4x_3^2 - 2x_3$

FOC

$$\partial y/\partial x_1 = 2x_1 - 2x_2 + 2x_3 = 0$$
$$\partial y/\partial x_2 = -2x_1 + 4x_2 = 0$$
$$\partial y/\partial x_3 = 2x_1 + 8x_3 - 2 = 0$$

When solved simultaneously, the system gives $x_1 = -1$, $x_2 = -0.5$, $x_3 = 0.5$. That is, there is a turning point at (-1, -0.5, 0.5).

SOC: When all the 2nd order and cross partial derivatives of the function are derived and evaluated at the turning point, the following (3x3) Hessian matrix describes the curvature of the function at the turning point.

$$H = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 4 & 0 \\ 2 & 0 & 8 \end{pmatrix}$$

The values of its successive principal minors are:

$$|H_1| = 2, \quad |H_2| = \begin{vmatrix} 2 & -2 \\ -2 & 4 \end{vmatrix} = 4, \ |H| = 16$$

Since they are all positive, i.e. the Hessian is positive definite, thus the turning point is a minimum.

Total Differentials

For a function of two or more independent variables, the *total differential* measures the change in the dependent variable brought about by a small change in each of the independent variables. If $z = f(x, y)$, the total differential dz is expressed mathematically as

$$\boxed{dz = z_x dx + z_y dy}$$

where z_x and z_y are the partial derivatives of z with respect to x and y respectively, and dx and dy are small changes in x and y. The total differential can be found by taking the partial derivatives of the function with respect to each independent variable and substituting these values in the formula above.

Example: Given the function $z = \ln(x^2 + 2y^3)$, the total differential is found as follows:

$$z_x = \frac{2x}{x^2 + 2y^3}$$

$$z_y = \frac{6y^2}{x^2 + 2y^3}$$

which, when substituted into the total differential formula, gives:

$$dz = \left(\frac{2x}{x^2 + 2y^3}\right) dx + \left(\frac{6y^2}{x^2 + 2y^3}\right) dy$$