FINA 1082 Financial Management

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Lecture 7
Modern Portfolio Theory II

November, 9, 2011

Objectives

- Examine the risk and expected return of two asset portfolios for different correlation levels

- Examine the risk and return of portfolios with many securities

- Analyze the limitations to risk diversification benefits

Risk-Return Tradeoff

Recall that a two-asset portfolio's expected return is

$$\mu_p = w_A\mu_A + w_B\mu_B$$

Alternatively,

$$\mu_p = w\mu_A + (1 - w)\mu_B$$

w = proportion of funds invested in security A

1 - w = proportion of funds invested in security B

The portfolio's return variance is

$$\sigma_P^2 = w^2\sigma_A^2 + (1 - w)^2\sigma_B^2 + 2w(1 - w)\sigma_A\sigma_B\,\rho_{AB}$$

We consider the following three cases

Case 1: Asset correlation is +1

Case 2: Asset correlation is -1

Case 3: Asset correlation is between +1 and -1

Risk-Return Tradeoff

Case 1: Assume $\rho_{AB} = +1$

There are no gains from diversification in this case

Combinations of the two securities lie on a straight line

$$\mu_p = w\mu_A + (1 - w)\mu_B \qquad (1)$$

$$\sigma_P^2 = w^2\sigma_A^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\sigma_A\sigma_B(+1)$$

$$[w\sigma_A + (1 - w)\sigma_B]^2$$

So,

$$[w\sigma_A + (1 - w)\sigma_B]^2 \qquad (2)$$

Solving (1) and (2) and eliminating w we get

$$\mu_p = \mu_B + (\sigma_p - \sigma_B)(\mu_A - \mu_B)/(\sigma_A - \sigma_B)$$

Example:

Assume $\mu_A = 14\%$, $\mu_B = 10\%$, $\sigma_A = 20\%$, $\sigma_B = 12\%$ and $\rho_{AB} = +1$. What is the relationship between risk and return in this case?

$$\mu_P = 10 + [(\sigma_P - 12)(14-10)]/(20 -12) = 4 + 0.5\sigma_P$$

Risk-Return Tradeoff

Case 2: Assume $\rho_{AB} = -1$

It is always possible to construct a zero risk portfolio in this case

$$\mu_p = w\mu_A + (1 - w)\mu_B \qquad (1)$$

$$\sigma_P^2 = w^2\sigma A^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\sigma_A\sigma_B(-1) = [w\sigma_A - (1-w)\sigma_B]^2$$

$$\text{So, } \sigma_p = w\sigma_A - (1 - w)\sigma_B \qquad (2)$$

In (2), set $\sigma_p = 0$ and solve for w to get

$w^* = \sigma_B /(\sigma_A + \sigma_B)$ (where w* is weight on A)

Example: $\mu_A = 14\%$, $\mu_B = 10\%$, $\sigma_A = 20\%$, $\sigma B = 12\%$ and $\rho_{AB} = -1$.

What is the minimum variance portfolio in this case?

$\sigma_p = 0$ when $w^* = 12/(20 + 12) = 0.375$ or 37.5%

$\mu_p = 0.375(14) + 0.625(10) =$ 11.5%

Risk-Return Tradeoff

Case 3: $-1 < \rho_{AB} < +1$

Some diversification benefits always exist

$\sigma_P^2 = w^2\sigma A_2 + (1-w)^2\sigma_B^2 + 2w(1-w)\sigma_A\sigma_B\ \rho_{AB}$

σ_P is minimized when w is

$$\boxed{w^*_{Min\,\sigma} = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B}}$$

Example: $\mu_A = 14\%$, $\mu_B = 10\%$, $\sigma_A = 20\%$, $\sigma_B = 12\%$, and $\rho_{AB} = 0$

Only when $\rho_{AB} = 0$, w* simplifies to

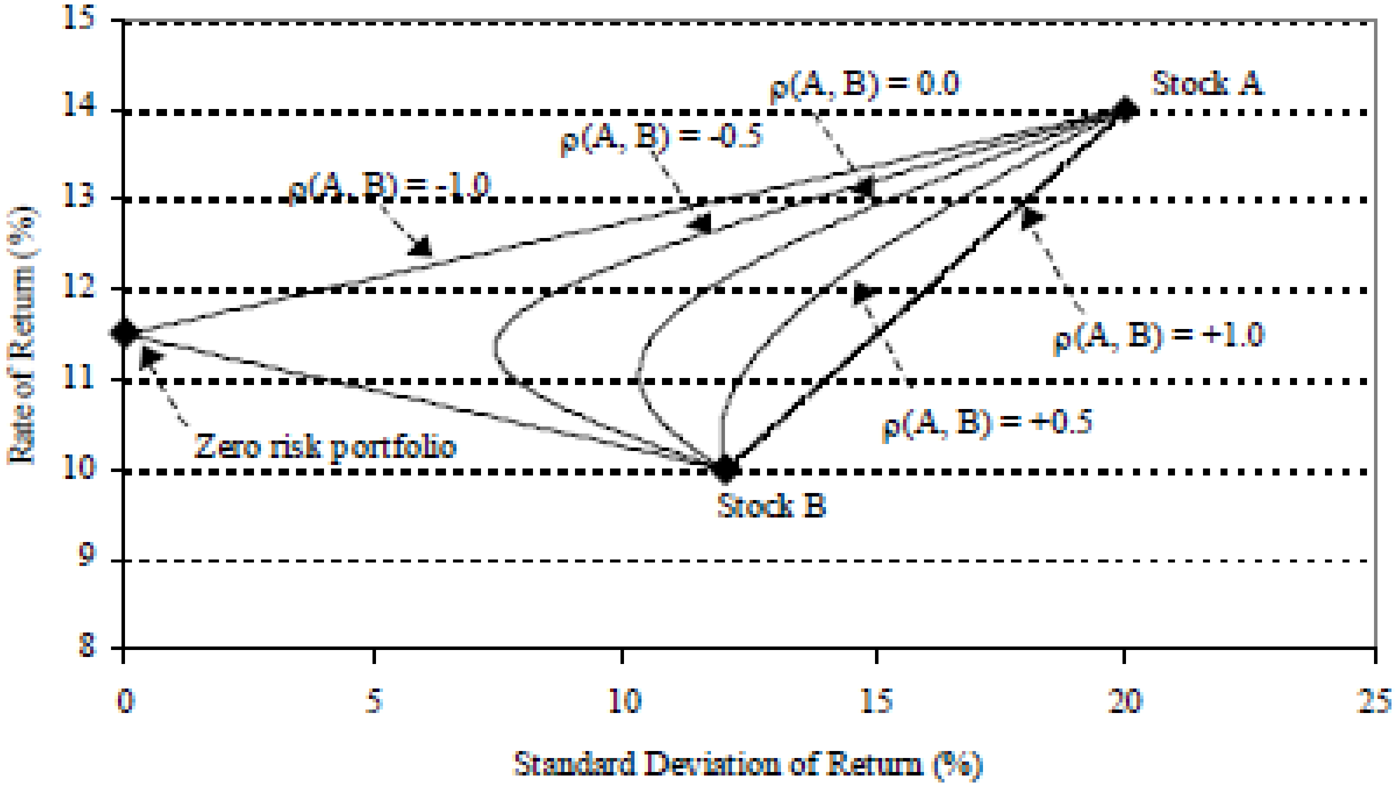
$$\boxed{w^*_{Min\,\sigma} = \sigma_B^2 / \left(\sigma_A^2 + \sigma_B^2\right)}$$

$W^*_{\min\sigma} = 12^2/(20^2 + 12^2) = 26.5\%$

$\sigma^*_p = 10.3\%$ and $\mu^*_p = 11.0\%$

Summary of Risk-Return Tradeoffs

Risk and expected return tradeoffs for different levels of correlation between returns



Portfolio Risk and Return: Many Assets

The expected return of an M asset portfolio is

$$E(r_p) = \sum_{j=1}^{m} w_j E(r_j)$$

The standard deviation of an M asset portfolio is

$$\sigma_p = \sqrt{\sum_{j=1}^{m} \sum_{k=1}^{m} w_j w_k \sigma_{jk}}$$

Note: $\sum_{j=1}^{m}$ is the summation operator - Sum all elements after the operator from $j = 1$ to $j = \text{m}$

$\sum_{j=1}^{m} \sum_{k=1}^{m}$ is a double summation operator - For $j = 1$ sum all the elements from $k = 1$ to $k = \text{m}$, for j = 2 sum all the elements from $k = 1$ to $k = \text{m}$, and so on until $j = \text{m}$.

Portfolio Risk and Return: Many Assets

$$\sigma_p = \sqrt{\sum_{j=1}^{m}\sum_{k=1}^{m} w_j w_k \sigma_{jk}} \quad = \text{Square root of sum of } \underline{\text{all}} \text{ elements in table}$$

$m = 2$

$$\begin{array}{cc} & \begin{array}{cccc} \text{Asset 1} & \text{Asset 2} & \text{Asset 3} & \text{Asset 4} \end{array} \\ \begin{array}{c} \text{Asset 1} \\ \text{Asset 2} \\ \text{Asset 3} \\ \text{Asset 4} \end{array} & \begin{bmatrix} w_1 w_1 \sigma_{11} & w_1 w_2 \sigma_{12} & w_1 w_3 \sigma_{13} & w_1 w_4 \sigma_{14} \\ w_2 w_1 \sigma_{21} & w_2 w_2 \sigma_{22} & w_2 w_3 \sigma_{23} & w_2 w_4 \sigma_{24} \\ w_3 w_1 \sigma_{31} & w_3 w_2 \sigma_{32} & w_3 w_3 \sigma_{33} & w_3 w_4 \sigma_{34} \\ w_4 w_1 \sigma_{41} & w_4 w_2 \sigma_{42} & w_4 w_3 \sigma_{43} & w_4 w_4 \sigma_{44} \end{bmatrix} \end{array} \quad m = 4$$

- Total number of SD terms = m = 4; Covariance terms = $m(m\text{-}1)$ = 12
  (*Note:* When m = 100, SD terms = 100; Covariance terms = 9,900!)
- $\sigma_{jk} = \rho_{jk}\, \sigma_j\, \sigma_k$
- ρ_{jk} = Correlation coefficient between returns for securities j and k
- σ_j , σ_k = Standard deviations for security j and k
- When $j = k$: $\rho_{jk} = 1$ and $\sigma_{jj} = \sigma_j^2$ (see diagonal terms)

Portfolio Risk and Return: 3 Assets

Example: You have invested 20% of your funds in security 1, 30% in 2 and the remainder in 3. What is the portfolio's expected return, variance and standard deviation of return?

			Correlations		
Security	Expected Return	Standard Deviation	1	2	3
1	10%	15%	1.0		
2	15%	10%	0.5	1.0	
3	20%	20%	-0.5	0.0	1.0

$E(r_p) = 0.2(0.10) + 0.3(0.15) + 0.5(0.20) = 0.165$ or 16.5%

Portfolio Risk and Return: 3 Assets

Covariance of returns, $\sigma_{jk} = \rho_{jk}\,\sigma_j\,\sigma_k$

$\sigma_{12} = (0.15)(0.10)(0.5) = 0.0075$
$\sigma_{13} = (0.15)(0.20)(-0.5) = -0.015$
$\sigma_{23} = (0.10)(0.20)(0.0) = 0$

	Asset 1	Asset 2	Asset 3
Asset 1	$(0.2)^2(0.15)^2$	$(0.2)(0.3)(0.0075)$	$(0.5)(0.2)(-0.015)$
Asset 2	$(0.2)(0.3)(0.0075)$	$(0.3)^2(0.1)^2$	$(0.5)(0.3)(0)$
Asset 3	$(0.5)(0.2)(-0.015)$	$(0.5)(0.3)(0)$	$(0.5)^2(0.2)^2$

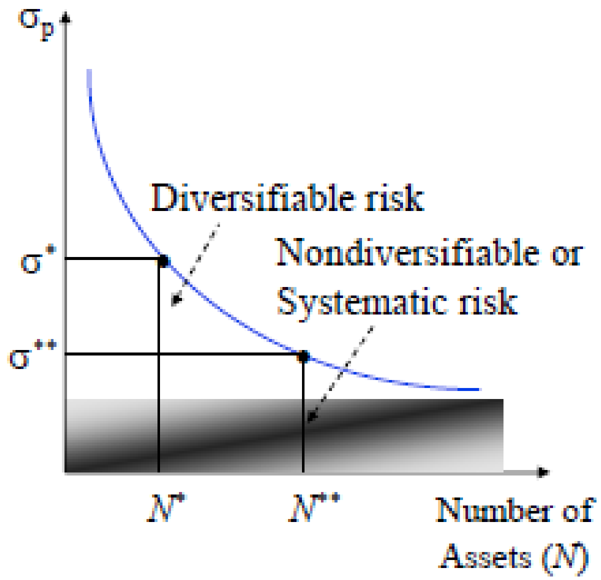
$$\sigma_p^2 = (0.2)^2(0.15)^2 + (0.2)(0.3)(0.0075) + (0.5)(0.2)(-0.015) + (0.3)(0.2)(0.0075) + (0.3)^2(0.1)^2 + (0.3)(0.5)(0) + (0.5)(0.2)(-0.015) + (0.5)(0.3)(0) + (0.5)^2(0.2)^2$$

$\sigma_p^{\ 2} = 0.0097$; $\sigma_p = (0.0097)^{1/2} = 0.0985$ or 9.85%

Note that 9.85% is lower than stand-alone risk of asset 2!

Limits to Diversification Benefits

As the number of securities in a portfolio increases, the risk of the portfolio decreases



Assume: $w_j = 1/N$, $\sigma_j = \sigma_k = \sigma$, $\rho_{jk} = \rho < +1$ for all j and k

Only in this case, we have:

$$\sigma_p^{\,2} = \sigma^2/N + [(N - 1)/N]\sigma_{jk}$$

As N increases, the first term approaches zero and the second term approaches σ_{jk}

Implication: The risk of large portfolios is determined by asset covariances

Limits to Diversification Benefits

Illustration: Consider portfolios of 10, 20, 50 and 100 stocks where you invest 1/N percent of your wealth in each stock, where each stock has a standard deviation of return of 10%, and where each pair of stocks has a correlation of 0.6.

a) Compute the standard deviations for these portfolios

b) What general relationship between portfolio standard deviation and the number of stocks is being illustrated here?

Limits to Diversification Benefits

Given: $w_j = 1/N$, $\sigma_j = 0.10$, $\rho_{jk} = +0.6$ for all j and k

$\sigma_{jk} = (0.1)(0.1)(0.6) = 0.006$

$\sigma_p{}^2 = \sigma^2/N + [(N - 1)/N]\sigma_{jk}$

$\sigma_p^2 = (0.1)2/N + [(N - 1)/N]0.006$

N = 10:	$\sigma_p = 0.0800$ or 8.00%	
N = 20:	$\sigma_p = 0.0787$ or 7.87%	(σ_p falls by 0.13%)
N = 50:	$\sigma_p = 0.0780$ or 7.80%	(σ_p falls by 0.07%)
N = 100:	$\sigma_p = 0.0777$ or 7.77%	(σ_p falls by 0.03%)
N = Huge:	$\sigma_p = 0.0775$ or 7.75%	(σ_p falls by 0.02%)

In large portfolios, asset covariances determine portfolio risk

As a portfolio becomes large in size its standard deviation falls, but at a declining rate

Key Concepts

- Diversification benefits exist as long as the correlation between two securities is less than perfect

- When two securities are perfectly negatively correlated it is always possible to form a zero risk portfolio

- Only diversifiable (unsystematic or idiosyncratic) risk can be eliminated via portfolio diversification

- The risk that cannot be eliminated via portfolio diversification is referred to as non-diversifiable (systematic or market) risk

- In large portfolios, asset covariances determine portfolio risk

- As a portfolio becomes large in size its standard deviation falls, but at a declining rate

Objectives

- Examine the risk-return tradeoff with a risky portfolio and riskfree asset
- Examine the separation theorem and its implications
- Define the market portfolio
- Develop the capital market line and its implications

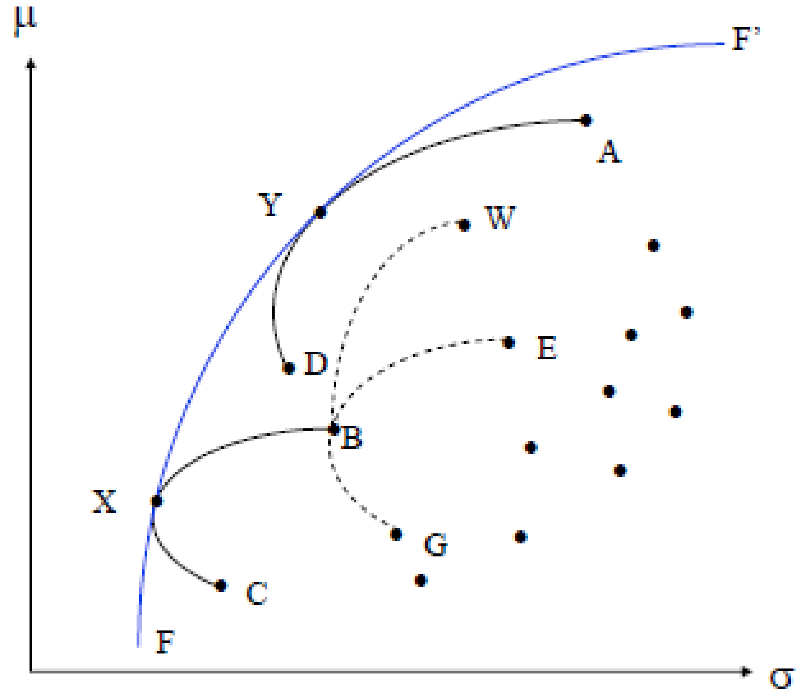
Risk-Return Frontiers with Many Assets

The efficient frontier FF' is an envelope of individual risk-return Frontiers

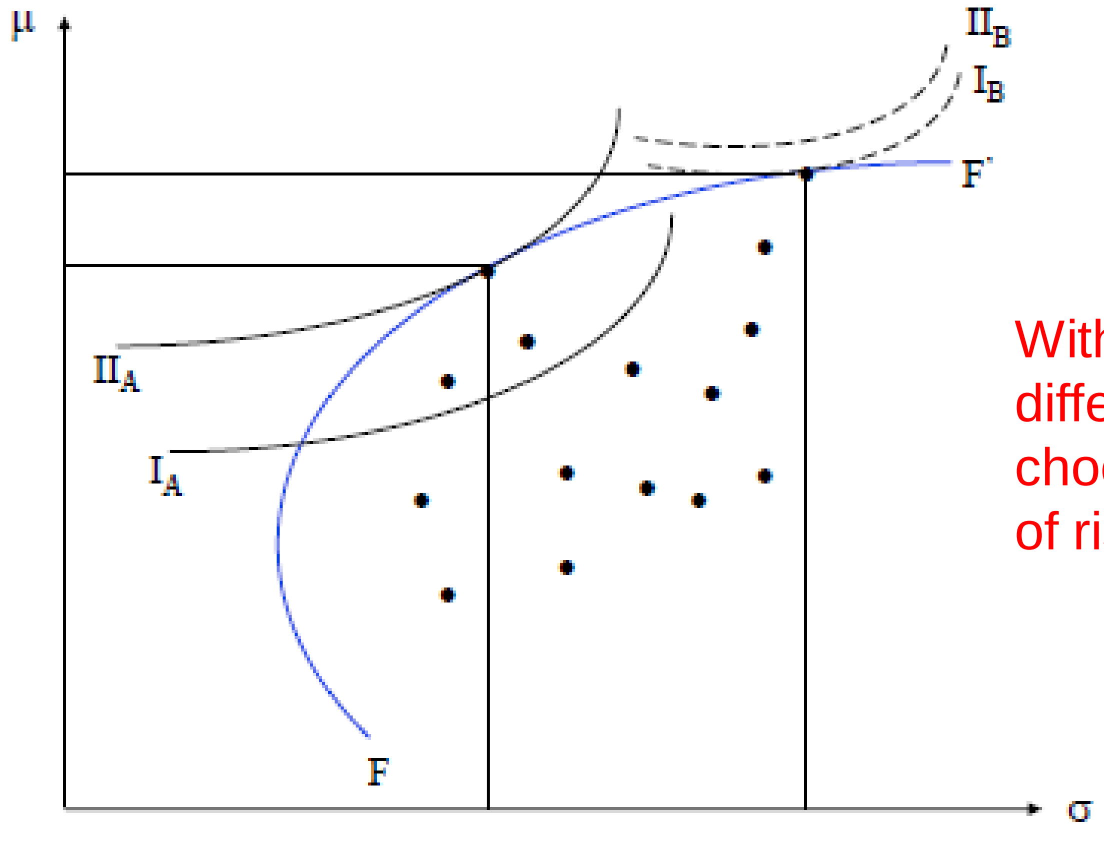
X is a portfolio of portfolios B and C and Y is a portfolio of A and D

FF' plots risky portfolios which have the lowest risk (σ) for a given expected return (μ)

Note: Portfolio W dominates E (higher m; lower s), but E may still be included in a portfolio because of low (or negative) covariance effects



Investor Choice Without a Riskfree Asset



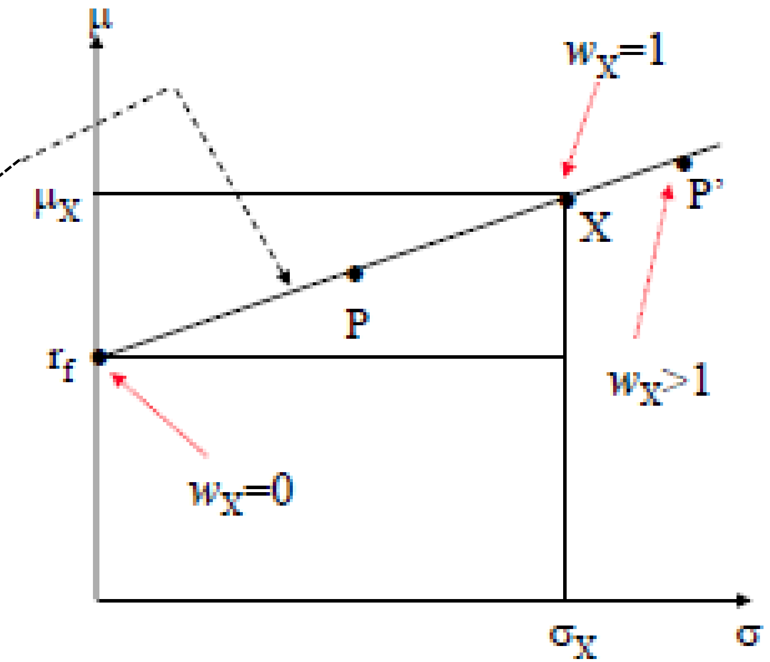
Without a riskfree asset different investors will choose different portfolios of risky assets

Introducing a Riskfree Asset

Assume you invest a proportion w_X in a risky portfolio X and $(1 - w_X)$ in the riskfree asset (r_f)

$\mu_p = w_X \mu_X + (1 - w_X) r_f$ (1)
$\sigma_p = w_X \sigma_X$ (2)
($\sigma_f = 0$ and $\sigma_{Xf} = 0$)

Eliminating w, we get:
$\mu_p = r_f + \sigma_p (\mu_X - r_f)/\sigma_X$



Introducing a Riskfree Asset

Class Exercise 1: You can invest \$10,000 in a riskfree asset earning 5% and a large risky portfolio X earning 15% with a standard deviation of 20%. What is the risk and return profile of investing \$5,000 in the riskfree asset and the risky portfolio, or borrowing \$5,000 to invest in the risky portfolio?

Case 1: Invest \$5,000 in the riskfree asset (ie., lend funds at the riskfree rate) and portfolio X

$w_f = w_X = 5000/10000 = 0.5$

$\mu_p = 0.5(0.05) + 0.5(0.15) = 10\%$

$\sigma_p = w_X \sigma_X = 0.5(0.20) = 10\%$

Case 2: Invest \$15,000 in portfolio X by borrowing \$5,000 at the riskfree rate of 5%

$w_X = 15000/10000 = 1.5$

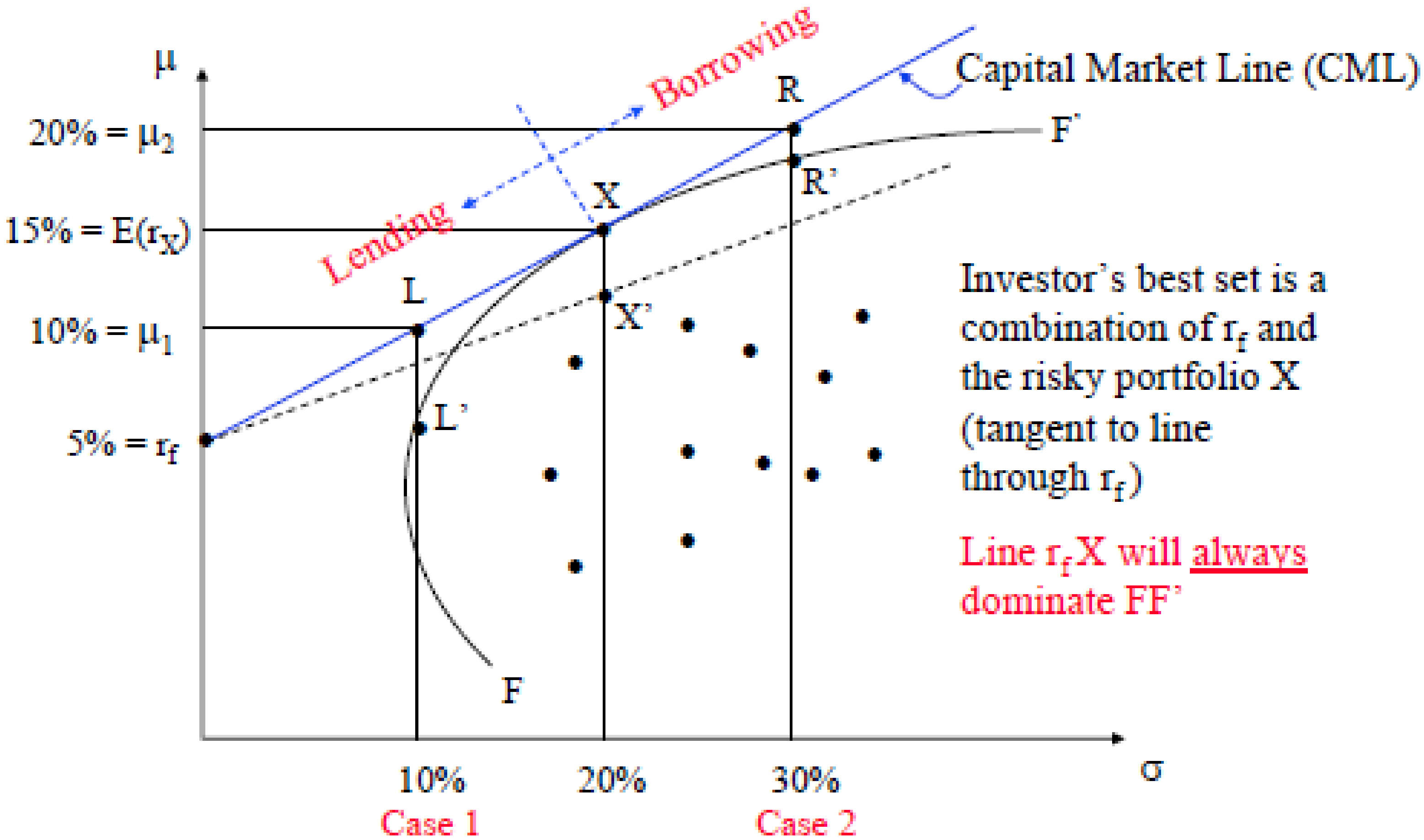
$w_f = -5000/10000 = -0.5$

$\mu_p = -0.5(0.05) + 1.5(0.15) = 20\%$

$\sigma_p = w_X \sigma_X = 1.5(0.20) = 30\%$

Note: Weights must add up to 1.0

Investor Choice With a Riskfree Asset



Investor's best set is a combination of r_f and the risky portfolio X (tangent to line through r_f)

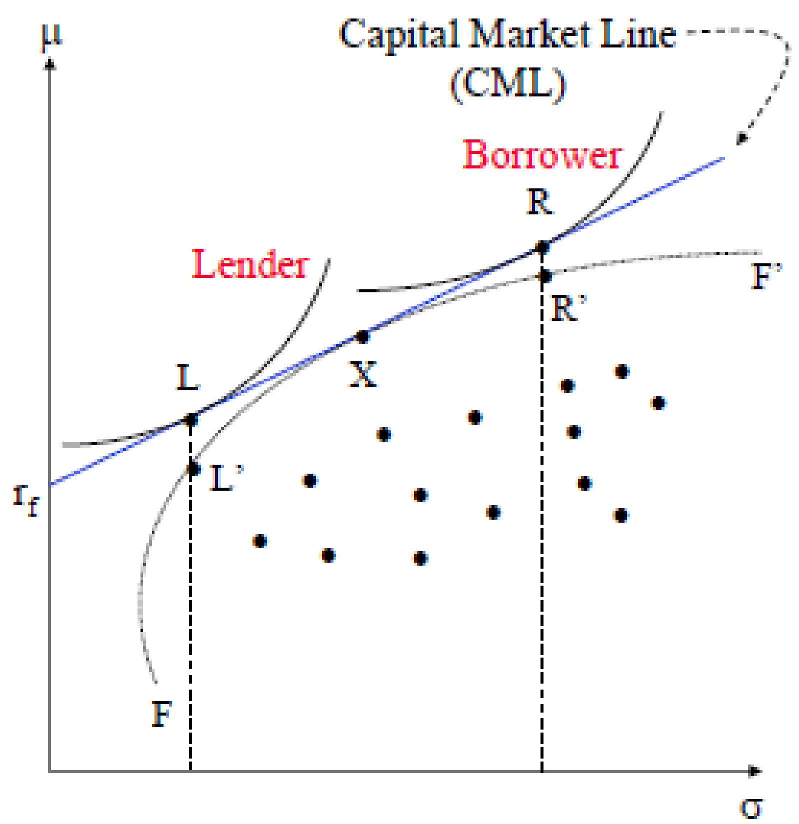
Line r_fX will always dominate FF'

The Separation Theorem

- Investors L and R maximize their utility by choosing a portfolio of rf and X, regardless of their individual risk preferences
  - L invests about 60% of wealth in rf and rest in X
  - R borrows 50% of wealth at rf and invests 150% in X

- The composition of the risky portfolio is separate from the riskfree/risky asset choice

- Line through rf and X is the capital market line (CML) which now dominates FF'



The Market Portfolio

In equilibrium, portfolio X must be the (value-weighted) market portfolio of all risky assets

Why is X the market portfolio?

- Suppose Stock A is 5% of total of risky assets by market value

- Suppose that only 2% of portfolio X is composed of Stock A

- 3/5th of A is not held by anyone
    - Stock A is in excess supply
    - The price of Stock A will fall, and its expected return will rise so investors will hold the remaining 3/5th of A in their (risky) portfolio

In equilibrium, X must be the market portfolio (now denoted as M)

The Capital Market Line

- The capital market line (CML) can only be used to “price” efficient portfolios

- The CML assumes portfolios are fully-diversified and efficient with zero unsystematic risk

- An efficient portfolio has the lowest level of risk for given level of expected return

- The CML cannot be used to price individual securities because they are not efficient and have unsystematic (firm specific) risk

The Capital Market Line

The CML's equation is

$$E(r_p) = rf + [E(r_m) - r_f\,][\sigma_p\,/\sigma_m]$$

The efficient portfolio's expected return equals the riskfree rate plus a risk premium where the risk premium equals the market risk premium [= E(rm) - rf] times the portfolio's proportionate risk [= σp /σm]

The CML's equation be rewritten as

$$E(r_p) = r_f + \{[E(r_m) - r_f\,]/\sigma_m\}\sigma_p$$

Intercept of CML = r_f, the riskfree rate
Slope of CML = $[E(r_m) - r_f\,]/\sigma_m$, expected return on the market portfolio (in excess of the riskfree rate) per unit of the market portfolio's total risk

Using the Capital Market Line

Class Exercise 2: You want to invest in a fully-diversified portfolio. The riskfree return is 5% and the market portfolio's expected return is 15% and its standard deviation of return is 20%. You have the following target risk levels (standard deviation of returns): σ_p = 15% and σ_p = 25%

a) How would you achieve the target risk levels? What are the expected returns of the two portfolios?

b) What is the CML's equation? Redo part (a) using the CML's equation

Answer to Class Exercise 2

a) Target risk level, σ_p = 15% - Lend at riskfree rate because your target level is below market risk; invest remaining funds in the market portfolio

$\sigma_p = 0.15 = w_m\sigma_m = w_m(0.20)$
$w_m = 0.15/0.20 = 0.75$ and $w_f = 1 - w_m = 0.25$
$E(r_p) = 0.25(0.05) + 0.75(0.15) = 12.5\%$

Target risk level, σ_p = 25% - Borrow at riskfree rate because your target level is above market risk; invest all the funds in the market portfolio

$\sigma_p = 0.25 = w_m\sigma_m = w_m(0.20)$
$w_m = 0.25/0.20 = 1.25$ and $w_f = 1 - w_m = -0.25$
$E(r_p) = -0.25(0.05) + 1.25(0.15) = 17.5\%$

b) CML's intercept, $r_f = 0.05$
CML's slope = $[E(r_m) - r_f]/\sigma_m = [0.15 - 0.05]/0.20 = 0.5$
CML's equation: $E(r_p) = 0.05 + 0.5\sigma_p$

Recap from Portfolio Analysis

- Separation Theorem: When riskfree borrowing and lending are possible, the decision of which portfolio of risky assets to hold is separated from the decision of how much risk to assume
    - The optimal portfolio involves a mix of the risky portfolio M and the riskfree asset

- Market Portfolio: Comprises all risky assets with weights equal to their share of total market capitalization – M changes as individual security prices change

- Capital Market Line (CML): When investors can borrow or lend at the riskfree interest rate, the minimum variance (efficient) portfolio lies on the tangent from r_f to the efficient frontier of risky portfolios

Question:

A risk-averse investor owns a stock portfolio worth $1 million. The investor believes that over the next year, this portfolio will rise in value by 20 percent or fall by 10 percent, with each outcome being equally likely. An investment bank offers the investor the following proposition: One year from today, the bank will pay the investor $100,000 if the portfolio value has decreased, but if the portfolio value goes up the investor receives nothing. In return, the investor must pay the investment bank $50,000 today. Does the investor accept the deal?

Example: Riskless Borrowing and Lending

Risky asset X → Three possible returns: -10%; 10%; 30% → Expected return = 10%; Standard deviation =16.3%

Risk-free asset Y → Return: 6% → Buying asset Y = Lending money at 6% interest

How would a portfolio with $100 (50%) in asset X and $100 (50%) in asset Y perform?

$100 Asset X
$100 Asset Y → Three possible returns: -2%; 8%; 18% → Expected return = 8%; Standard deviation =8.16%

Portfolio has lower return but also less volatility than 100% in X
Portfolio has higher return and higher volatility than 100% in risk-free

Riskless Borrowing and Lending (Continued)

What if we sell short asset Y instead of buying it?
Borrow \$100 at 6% Must repay \$106

Invest \$300 in X
Original \$200 investment plus \$100 in borrowed funds

When X Pays –10%

$$\text{Net Return on \$200 Investment} = \frac{\$270 - \$106 - \$200}{\$200} = -18\%$$

When X Pays 10%

$$\text{Net Return on \$200 Investment} = \frac{\$330 - \$106 - \$200}{\$200} = 12\%$$

When X Pays 30%

$$\text{Net Return on \$200 Investment} = \frac{\$390 - \$106 - \$200}{\$200} = 42\%$$

Expected return on the portfolio is 12%. Higher expected return comes at the expense of greater volatility

Riskless Borrowing and Lending (Continued)

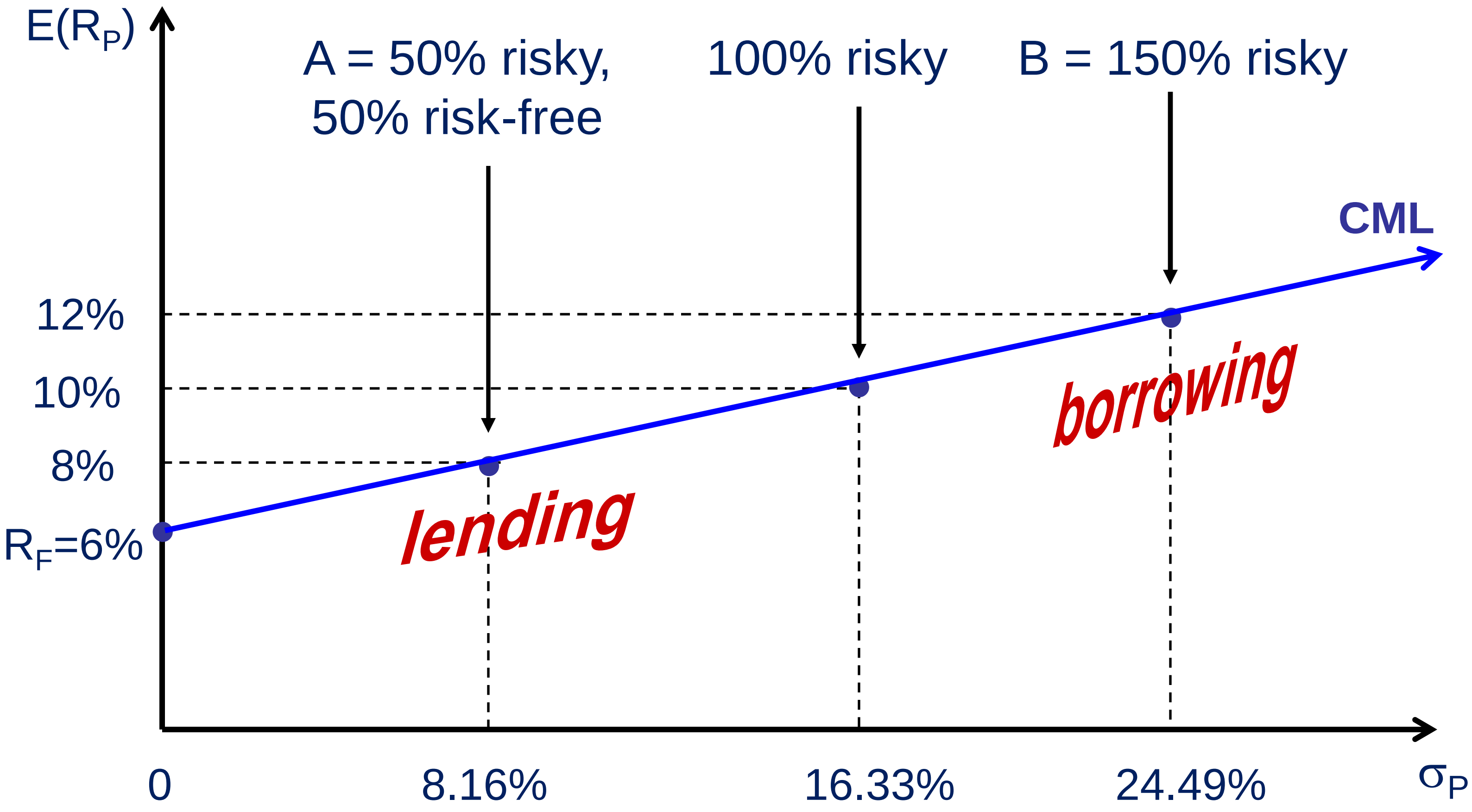
The more we invest in X, the higher the expected return

The expected return is higher, but so is the volatility

This relationship is linear

Portfolio	Expected Return	Standard Deviation
50% risky, 50% risk-free	8%	8.16%
100% risky, 0% risk free	10%	16.33%
150% risky, -50% risk free	12%	24.49%

Portfolios of Risky & Risk-Free Assets



Key Concepts

• Without a riskfree asset, the choice of the “best” risky portfolio depends on investors’ attitude towards risk

• With a riskfree asset, the choice of the “best” risky portfolio is separate from investors’ attitude towards risk

• The “optimal” risky portfolio is an efficient portfolio comprising all assets trading in the market

• The line passing through the riskfree asset and the market portfolio is the capital market line

• All investors invest their funds in the risky market portfolio and then borrow or lend to move along the CML depending on their attitude towards risk