Portfolio Management

Session 2

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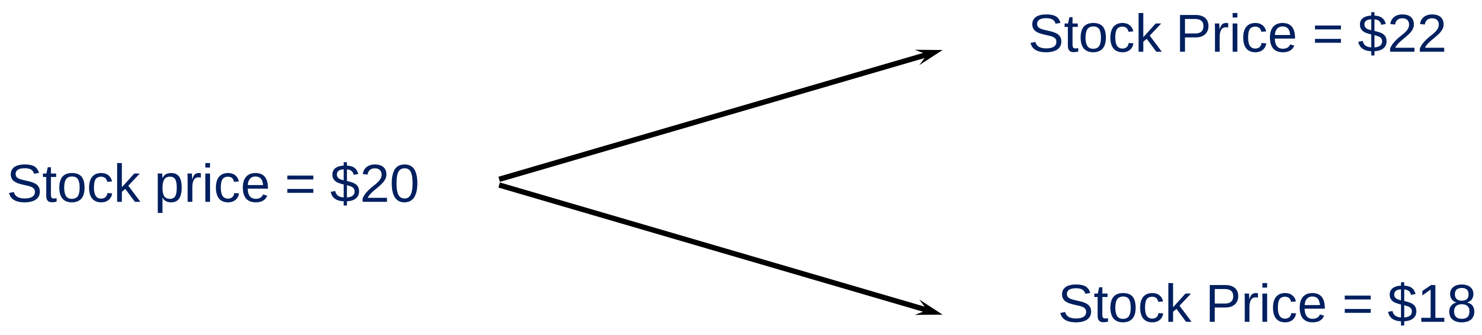
The Pricing of Options

Binomial option-pricing model

This model recognizes that investors can combine options (either calls or puts) with shares of the underlying asset to construct a portfolio with a risk-free payoff

A Simple Binomial Model

- A stock price is currently $20
- In three months it will be either $22 or $18

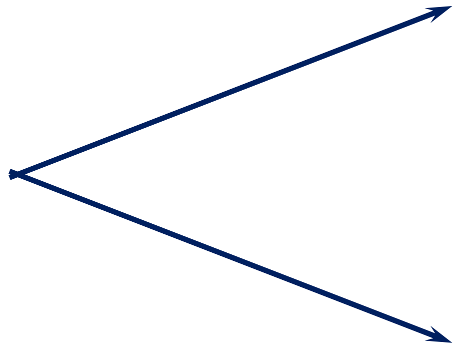


A Call Option

A 3-month call option on the stock has a strike price of 21.

Stock Price = $22
Option Price = $1

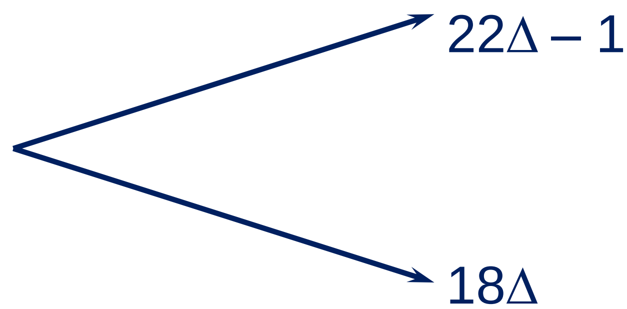
Stock price = $20
Option Price=?



Stock Price = $18
Option Price = $0

Setting Up a Riskless Portfolio

- Consider the Portfolio: long Δ shares
  short 1 call option



- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$

Valuing the Portfolio
(Risk-Free Rate is 12%)

- The riskless portfolio is:

  long 0.25 shares
  short 1 call option

- The value of the portfolio in 3 months is

$$22 \times 0.25 - 1 = 4.50$$

- The value of the portfolio today is

$$4.5e^{-0.12 \times 0.25} = 4.3670$$

Valuing the Option

- The portfolio that is
  - long 0.25 shares
  - short 1 option

  is worth 4.367
- The value of the option is therefore

  0.25 x 20 – c = 4.367

  c = 0.633 (= 5.000 – 4.367)

Generalization

A derivative lasts for time *T* and is dependent on a stock

$$\begin{tikzcd} & \begin{matrix} S_0u \\ f_u \end{matrix} \\ \begin{matrix} S_0 \\ f \end{matrix} \arrow[ur] \arrow[dr] & \\ & \begin{matrix} S_0d \\ f_d \end{matrix} \end{tikzcd}$$

Generalization (continued)

- Consider the portfolio that is long Δ shares and short 1 derivative

$$S_0 u\Delta - f_u$$

$$S_0 d\Delta - f_d$$

- The portfolio is riskless when

$S_0 u\Delta - f_u = S_0 d\Delta - f_d$ or $\Delta = \dfrac{f_u - f_d}{S_0 u - S_0 d}$

Generalization (continued)

- Value of the portfolio at time T is

$$S_0 u\Delta - f_u$$

- Value of the portfolio today is

$$(S_0 u\Delta - f_u)e^{-rT}$$

- Another expression for the portfolio value today is

$$S_0\Delta - f$$

- Hence: $$f = S_0\Delta - (S_0 u\Delta - f_u)e^{-rT}$$

Generalization (continued)

- Substituting for Δ we obtain

$$f = [\, p f_u + (1 - p) f_d \,] e^{-rT}$$

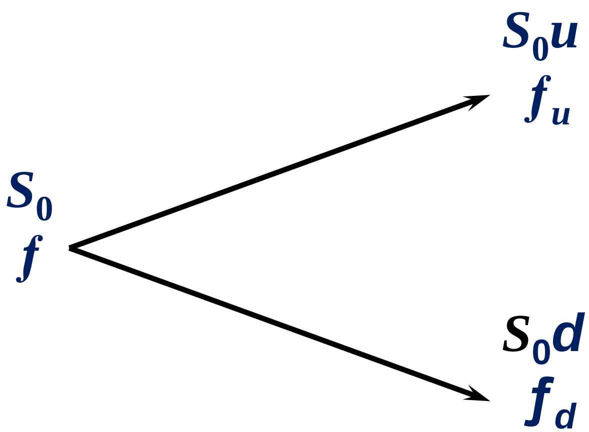
where

$$p = \frac{e^{rT} - d}{u - d}$$

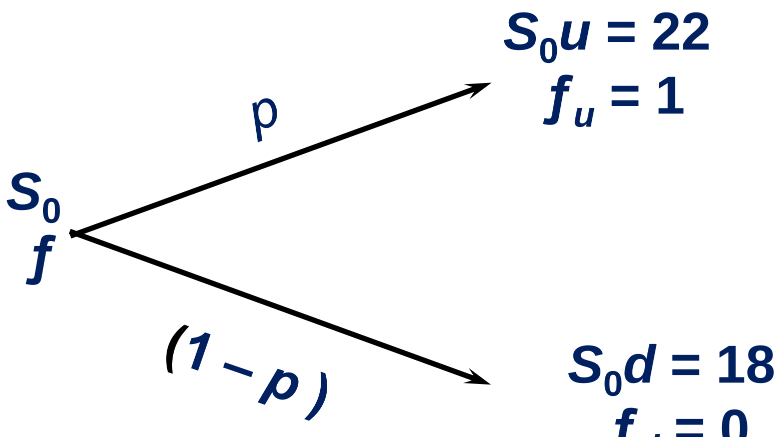
Risk-Neutral Valuation

- *p* and *(1-p)* can be interpreted as risk-neutral probabilities of up and down movements

- The value of a derivative is then its expected payoff in a risk-neutral world discounted at the risk-free rate



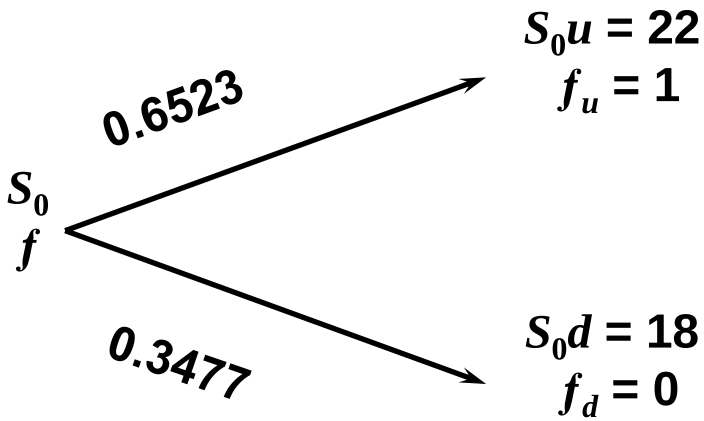
Original Example Revisited



- we can use the formula

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12\times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

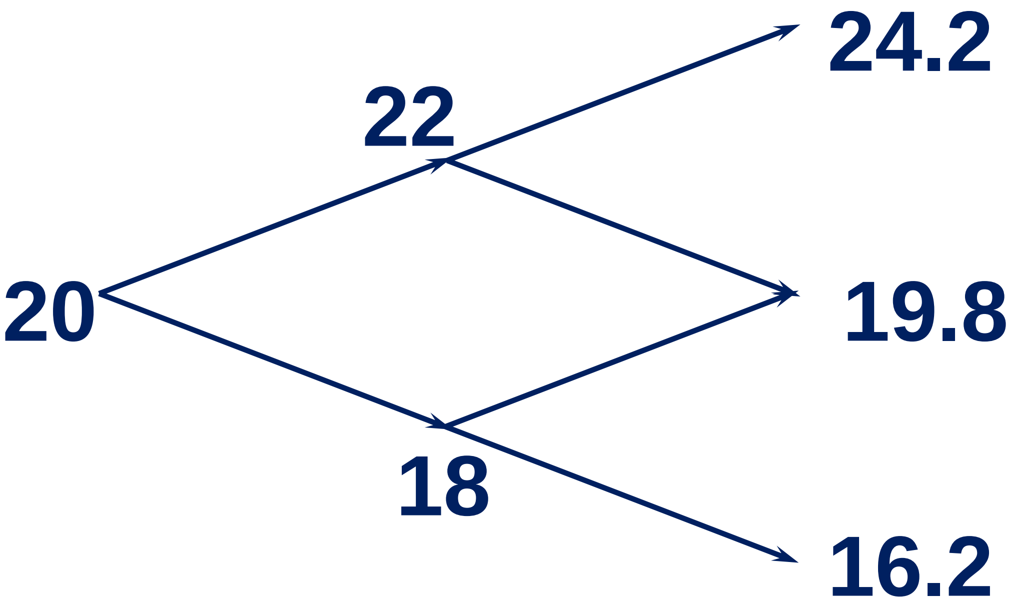
Valuing the Option



The value of the option is

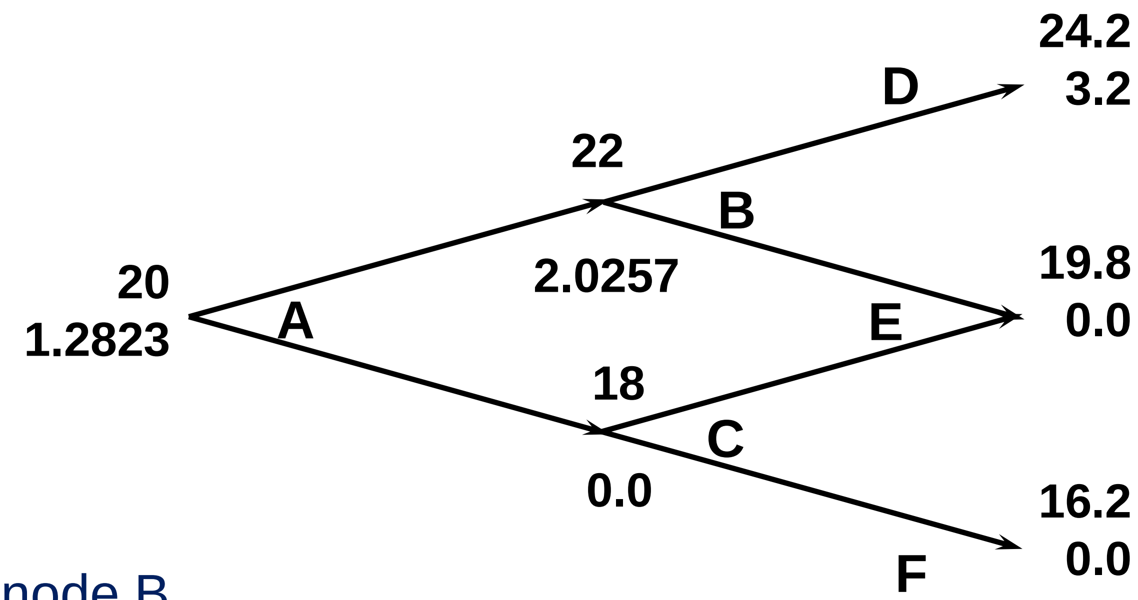
$$e^{-0.12 \times 0.25}(0.6523 \times 1 + 0.3477 \times 0) = 0.633$$

A Two-Step Example



- Each time step is 3 months
- K=21, r=12%

Valuing a Call Option



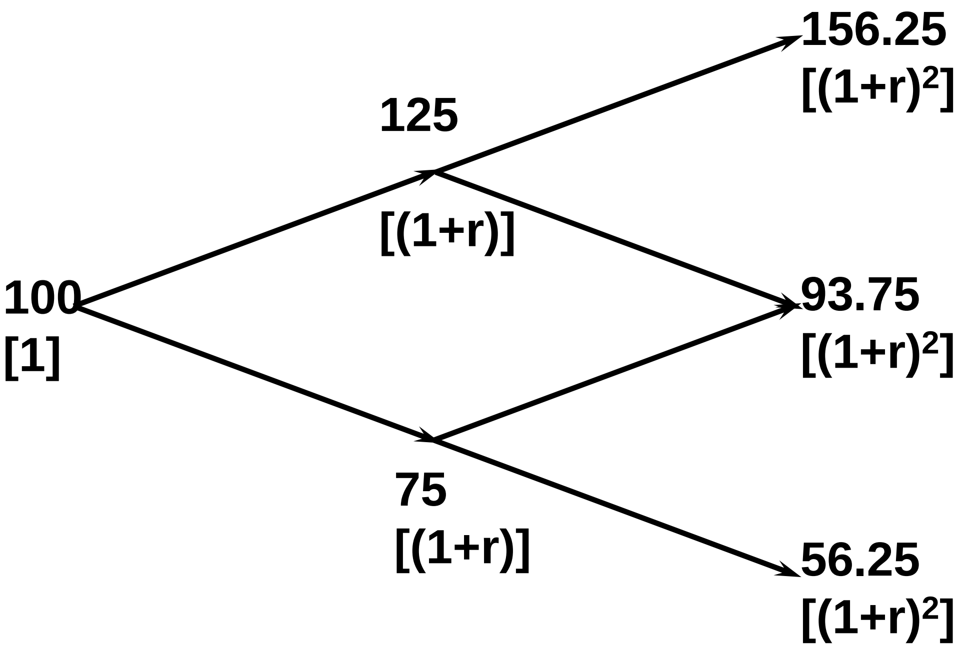
- Value at node B

  $= e^{-0.12 \times 0.25}(0.6523 \times 3.2 + 0.3477 \times 0) = 2.0257$

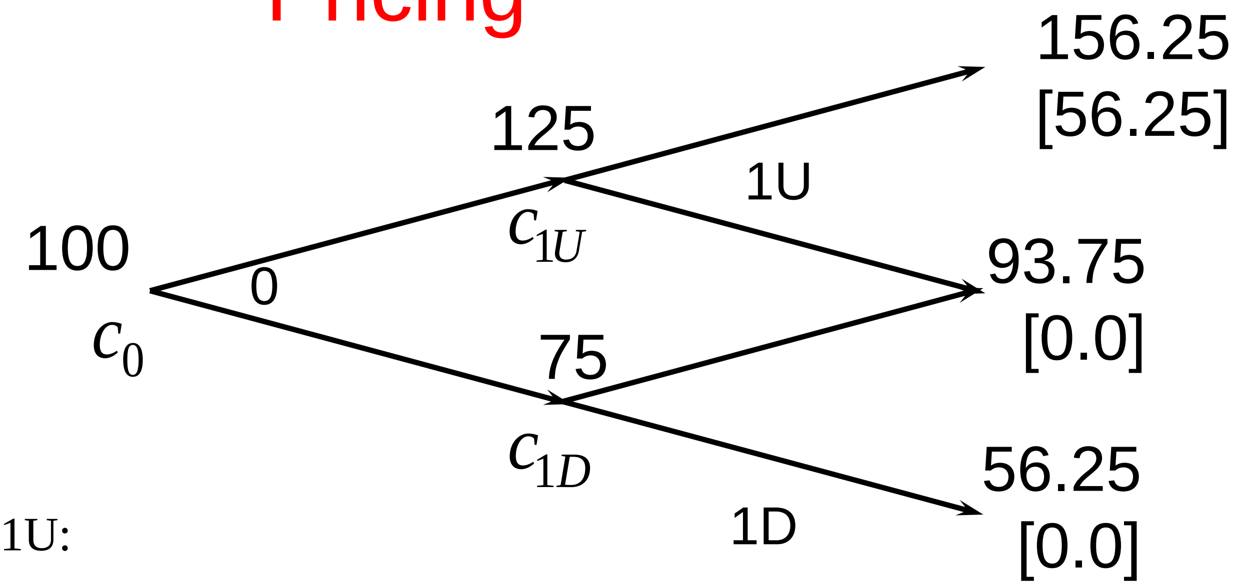
- Value at node A

  $= e^{-0.12 \times 0.25}(0.6523 \times 2.0257 + 0.3477 \times 0) = 1.2823$

Extension to two periods



Pricing



- Price at knot 1U:

$$c_{1U} = \frac{1}{1.06}\left[\frac{1.06-0.75}{0.5}56.25+0\right] = 32.9$$

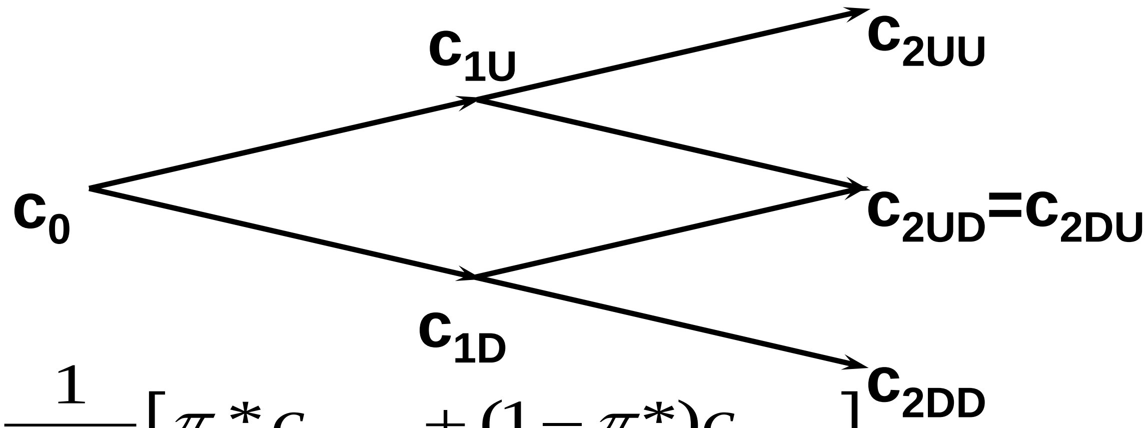
- Price at knot 1D:

$$c_{1D} = \frac{1}{1.06}[0+0] = 0.0$$

- Price at knot 0:

$$c_0 = \frac{1}{1.06}\left[\frac{1.06-0.75}{0.5}32.9+0\right] = 19.24$$

Binomial Generalisation Two Periods



$$c_{1U} = \frac{1}{1+r}\left[\pi^* c_{2UU} + (1-\pi^*)c_{2UD}\right]$$

$$c_{1D} = \frac{1}{1+r}\left[\pi^* c_{2DU} + (1-\pi^*)c_{2DD}\right]$$

$$c_0 = \frac{1}{(1+r)}\left[\pi^* c_{1U} + (1-\pi^*)c_{1D}\right]$$

$$\Rightarrow c_0 = \frac{1}{(1+r)^2}\left[\pi^{*2} c_{2UU} + 2\pi^*(1-\pi^*)c_{2UD} + (1-\pi^*)^2 c_{2DD}\right]$$

What happens with the Binomial Model when we act such that the number of periods between today and $T \rightarrow \infty$ (the length of each interval goes to zero or continuous trading)?

We obtain the famous *Black-Scholes Option Pricing Formula* for European calls**:**

$$c_0 = P_0 N(d_1) - K e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(P_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}};\ d_2 = d_1 - \sigma\sqrt{T}$$

where *N(d)* is the cumulative probability distribution function for a standardized Normal distribution

It is the present value of the weighted difference between the benefits and costs of exercising the option, where the weights are some special probabilities

and the price of *puts*?

- It can be shown that:

$$p_0 = Ke^{-rT}N(-d_2) - P_0 N(-d_1)$$

- Alternatively, we could use the *put-call parity relationship* (a call and a put on the same underlying and same exercise price and maturity):

$$c_0 + Ke^{-rT} = p_0 + P_0$$

$$\Rightarrow p_0 = c_0 - (P_0 - Ke^{-rT})$$

The intuition behind the Black-Scholes option pricing formula

- ❑ Mathematical model that accurate pricing options

- ❑ Model from 1973, first appears as Black-Scholes option Pricing Model (Fisher Black and Myron Scholes in Journal of Political Economy)

- ❑ Same year Robert Merton published the Theory of Rational Option Pricing

- ❑ These two articles together provided the foundation for Option Pricing

Basic underlying Intuition

- Remarkable simple to price an option when there is no time to expiration

- Call option that gives the right to buy 100 shares of IBM at $200 and if the price of IBM is $215, the option must worth $15, so the all contract worth $1,500

- Price of the option at the expiration is the underlying stock price S minus the exercise price X (*no need of mathematical model*)

Real Question: What about before expiration

- Model of Black and Scholes and Merton comes in

- Clear what makes the option payoff: The option payoff if the underlying asset achieves a final value above the exercise price.

How can we put this in a pricing formula?

- ❑Similarities with the value of a call option at expiration

- ❑ At expiration S – X = $15 or zero

- ❑ In the BS Model one of S (current stock price) but not X. Instead we have the present value of the exercise or strike price.

- ❑ We need to mathematically determined what the probability is that the call option finish *in the money*. What is the probability that the underlying commodity price exceeds the exercise price by the time of maturity?

- ❑ They made assumptions how the stock prices will behave. Stock returns following a certain type of process. Normal distribution. The returns with normal distribution and there is a certain mean return that will expect and there is a volatility.

❑ With that information given what we expect in terms of the mean or average return and volatility we can actually calculated what the probability is that the stock price finishes above a certain cut-off point.

❑ We can calculate that in a very simple way using the cumulative normal distribution. And these are the inputs for the BS formula.

❑So using in particular: the volatility of the underlying asset, we can calculate the probability that the underlying stock price finishes in the money. And this is the essential ingredient for the BS formula

❑If one look to the formula we can see that the essential things are there:
1) stock price minus the present value of the exercise price (we are not at the expiration we are today, and everything needs to be adjusted to today’s values),

2) The cumulative normal distribution which give us sense of how likely the stock price will be to exceeding the exercise price.

3) In the cumulative normal distribution the volatility of the underlying commodity return which is essential,

4) time to expiration

- ❑ All these determinants of option value appear in BS formula
- ❑The essential intuition is quite straightforward
- ❑ The option has value if there is a probability that the underlying commodity value will be above the exercise price
- ❑This is basically the foundation of the BS formula

Black and Scholes Formulas

$$c_0 = P_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(P_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}; \; d_2 = d_1 - \sigma\sqrt{T}$$

The Concepts Underlying Black-Scholes Model

- The option price and the stock price depend on the same underlying source of uncertainty
- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate
- This leads to the Black-Scholes differential equation

Assumptions Underlying Black-Scholes

- No dividends
- Underlying stock returns are normally distributed
- No transaction costs
- Risk free rate for lending and borrowing
- Volatility and interest rates are constant up to maturity

Black & Scholes in Practice

- *Forward Use*:

Underlying Price
Exercise Price
Time-to-Maturity
Risk-free Interest Rate
Estimated Volatility

Black-Scholes Formula

OPTION PRICE

- *Backward Use*:

Underlying Price
Exercise Price
Time-to-Maturity
Risk-free Interest Rate
Implied Volatility
(forward looking volatility)

Black-Scholes Formula

OPTION PRICE

Example of option valuation using Black-Scholes

What is the value of a European call option with an exercise price of 6.70 and a maturity date of 276 days from now if the current share price is $12.41, standard deviation is 25% p.a. and the risk-free rate is 10.50%. Assume there to be 365 days in a year.

Use the Black-Scholes formula to derive your result.

$$c_0 = P_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$d_1 = \frac{ln(P_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$d_1 = \frac{\left[ln\left(\frac{12.41}{6.70}\right)+\left[0.1050+\frac{1}{2}(0.25)^2\right]\left(\frac{276}{365}\right)\right]}{0.25\sqrt{\frac{276}{365}}}$$

$$d_1 = 3.3093$$

$$d_2 = 3.3093 - 0.25\sqrt{\frac{276}{365}} = 3.0919$$

$$N(d_1) = 0.9995$$

$$N(d_2) = 0.9990$$

$$C = 12.41 \times 0.9995 - 6.7 \times e^{-0.105 \times \left(\frac{276}{365}\right)} \times 0.9990$$

$$= 12.4038 - 6.1824 = \$6.2214\ (\$6.22)$$

Questions

1) In what sense can the terms N (d1) and N (d2) in the BS model be interpreted as probabilities that a call option will expire in the money?

2) The stock of Cloverdale Food Processors currently sells for $40. A European Call option on Cloverdale stock has an expiration date six months in the future and a strike price of $38. The estimate of the annual standard deviation of Cloverdale stock is 45 percent, and the risk-free rate is 6 percent. What is the call worth?

3) Using the following table, price one-year call options using different annual volatilities. The strike price is $30.00, with and underlying spot price of $30.00 and a risk-free rate of 3 percent APR.

Volatility	N(d1)	N(d2)	Call Price
20%	0.5987	0.5199	
25%	0.5968	0.4980	
30%	0.5987	0.4801	

Answer Question 2:

$$d1 = \frac{\left(\frac{40}{38}\right) + \left(0.06 + \frac{0.45^2}{2}\right)\frac{1}{2}}{0.45\sqrt{\frac{1}{2}}} = \frac{0.0513 + 0.0806}{0.3182} = 0.4146$$

$$d2 = d1 - \sigma\sqrt{t} = 0.4146 - 0.45\sqrt{\frac{1}{2}} = 0.0964$$

$$N(0.4146) + 0.6608$$

$$N(0.0964) = 0.5384$$

$$C = 40(0.6608) - 38\left(2.718^{-(0.,06)(0.5)}\right)(0.5384) = \$6.58$$

Implied Volatility

- The implied volatility of an option is the volatility for which the B&S price equals that of the market; the formula is invertible *one to one*

- Investors and analysts use this data to:

  - Approximate the risk the market assigns to the underlying

  - Price options with similar underlyings, but less liquidity

  - Predict future volatility

- Problem: Which implied volatility do we have to choose when it differs for options with different exercise prices?

Implied Volatility

- The value of σ obtained by “inverting” the BS equation to calculate the level of volatility implied by the option’s market price.

- Some options traders quote implied volatility sometimes rather than the price of the option

- It is important to understand what implied volatility is for an option.

- Basically the volatility that we all refers to is the volatility of the underlying commodity return. What is the volatility of Microsoft? Are the volatility of the returns for Microsoft.

- For option pricing we always look to continuous return so will be the log of the price today divided by the price yesterday, will be the daily return.

Implied volatility: Special measure

How we derive it?

- There is a market price that the options are traded and there is the BS model, and basically we got almost of the determinants of the BS. We now the stock price, the exercise price, time to expiration, the interest rate and also know if some dividends are going to be paid.

- We do not know the volatility.

- We can try different levels of volatility in BS Model until the BS Model delivers a price for the call option which is exactly equal to the price that we observe in the market. This volatility is the one called as "implied volatility"

- Very similar to what is done to YTM or IRR.

- Implied volatility comes from The BS Model which is a non linear function, the YTM comes from the bond pricing function which is a non linear function.

- We trail and error until we get the BS formula to deliver a price that is exactly equals to the market price of the call option.

- When we get that volatility the markets assessment of what the volatility will be in the future foe the underlying security.

- So just like the yield to maturity tell us something about what people expect interest rates to be in the future the BS implied volatility tell us about what people think about volatility of the underlying commodity return will be in the future

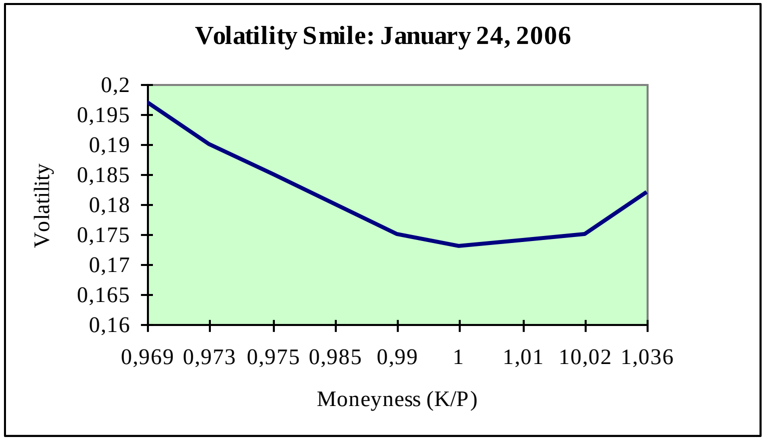
Questions

A six-month call option with a strike price of $25.00 is selling for $3.50. Assuming the underlying stock price is also $25.00 and the risk-free rate is 6 percent APR, use the following table to determine the volatility (i.e, standard deviation of the return) implied using the option price. (Hint: Price the option using the table to determine which volatility generates a price of $3.50)

Volatility	**N(d1)**	**N(d2)**
40%	0,5799	0,4859
45%	0.6000	0.4742
50%	0.6032	0.4634

Volatility Smiles

- Implied volatility and B&S: The *smiles*. The following graph is quite common (asymmetric smile):



- This suggests that the distribution of the underlying has:
  - Thicker tails than the normal distribution (excess kurtosis)
  - Left tail thicker than the right one (higher volatility when prices decrease or negative correlation between changes in volatility and changes in the underlying)

More uses of Options

- We underline three important applications of options:

- Pricing banks: insurance premium for the contribution of banks to the nation wide deposit's insurance

- Pricing firms' shares when capital structure contains both equity and debt (equity is a call option on the assets with exercise price being the nominal value of debt)

- Structured products (very common for guaranteed mutual funds)

Arbitrage & Put Call Parity

Since the payoff on a combination of a long call and a short put are equivalent to leveraged equity, the prices must be equal.

$$C - P = S_0 - X / (1 + rf)^T$$

If the prices are not equal arbitrage will be possible.

Disequilibrium Example

Stock Price = 110
Put Price = 5
Maturity = .5 yr

Call Price = 17
Risk Free = 10.25%
X = 105

$$C - P = S_0 - X / (1 + rf)^T$$

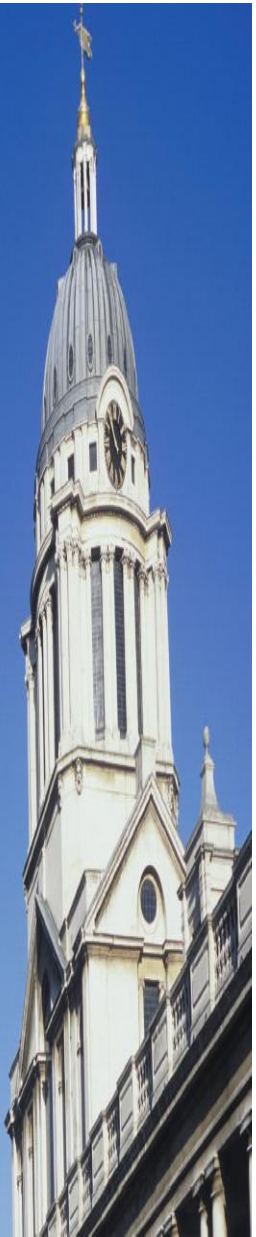
$$17 - 5 > 110 - (105/1.05)$$

$$12 > 10$$

Since the leveraged equity is less expensive, acquire the low cost alternative and sell the high cost alternative.

Put-Call Parity Arbitrage

Position	Immediate Cashflow	Cashflow in Six Months $S_T < 105$	$S_T \geq 105$
Buy Stock	-110	S_T	S_T
Borrow $X/(1+r)^T = 100$	+100	-105	-105
Sell Call	+17	0	$-(S_T-105)$
Buy Put	-5	$105-S_T$	0
Total	2	0	0



- How do we select a portfolio in practice?
- Alternative statistical bases
- Finding efficient risky portfolios

Recommended Reading

– On the financial modelling using, e.g., Excel:

- Benninga, S., *Financial Modeling*, USA: The MIT Press

– On portfolio optimisation within Excel:

- Bodie, Z., Kane, A. and Marcus, A.J., *Investments* (5th edition), Irwin/McGraw-Hill, 2002, Chapter 8

Recommended Reading (cont.)

– Journals:

- Harlow, W.V., 'Asset Allocation in a Downside-Risk Framework', *Financial Analysts Journal*, September-October 1991

- Sortino, F.A. and van der Meer, R., 'Downside risk', *Journal of Portfolio Management*, Summer 1991, Vol. 17, Iss. 4, pp 27-31

- Grootveld, H. and Hallerbach, W., 'Variance vs downside risk: Is there really that much difference?', *European Journal of Operational Research*, 1999, Vol. 114, pp 304-319

Recommended Reading (cont.)

- Sortino, F.A. and Forsey, H.J., ‘On the use and misuse of downside risk’, *Journal of Portfolio Management*, Winter 1996, Vol. 22, Iss. 2, pp 35-43
- Chopra, V.K. and Ziemba, W.T., ‘The Effect of Errors in Means, Variances and Covariances on Optimal Portfolio Choice’, *Journal of Portfolio Management*, Winter 1993, Vol. 19, Iss. 2, pp 6-11

Key elements

- **Risk Averse**

Describes an investor who, when faced with two investments with a similar expected return (but different risks), will prefer the one with the lower risk.

- **Asset Allocation**

The process of dividing a portfolio among major asset categories such as bonds, stocks or cash. The purpose of asset allocation is to reduce risk by diversifying the portfolio.

The ideal asset allocation differs based on the risk tolerance of the investor. For example, a young executive might have an asset allocation of 80% equity, 20% fixed income, while a retiree would be more likely to have 80% in fixed income and 20% equities.

Key elements

- Risk-Free Asset

An asset which has a certain future return. Treasuries (especially T-bills) are considered to be risk-free because they are backed by the U.S. government.

Because they are so safe, the return on risk free assets is very close to the current interest rate.

Many academics say that there is no such thing as a risk-free asset because all financial assets carry some degree of risk. Technically, this may be correct. However, the level of risk is so small that, for the average investor, it is OK to consider U.S. Treasuries or Treasuries from stable Western governments to be risk-free.

Top-Down Analysis

Capital Allocation Decision: choice of proportion of the overall portfolio to place in safe but low-return versus risky but higher-return securities.

Asset Allocation Decision: Distribution of risky investments across broad asset classes: Stocks, bonds, etc

Selection Decision: Choice of which particular securities to hold within each asset class

Allocating Capital Between Risky & Risk Free Assets

- It's possible to split investment funds between safe and risky assets.
- Risk free asset: proxy; T-bills
- Risky asset: stock (or a portfolio)

Allocating Capital Between Risky & Risk Free Assets (cont.)

Issues

- Examine risk/return tradeoff.
- Demonstrate how different degrees of risk aversion will affect allocations between risky and risk free assets.

Diversification and Risk

- Diversification and portfolio risk
  - Firm-specific risk: diversification can reduce risk to arbitrarily low levels
  - Risk that affects all firms: even extensive diversification cannot eliminate risk
  - Market (systematic) risk vs. unique (non-systematic) risk

- Portfolio of two risky assets
  - Two mutual funds (bond and stock portfolio)
  - Expected return on the two-asset portfolio is:

$$E(r_p) = w_d E(r_d) + w_e E(r_e)$$

Diversification and Risk (cont.)

– The variance of the two-asset portfolio is:

$$\sigma_p^2 = w_d^2\sigma_d^2 + w_e^2\sigma_e^2 + 2w_d w_e Cov(r_d, r_e)$$

– The variance of the three-asset portfolio is:

$$\sigma_p^2 = w_x^2\sigma_x^2 + w_y^2\sigma_y^2 + w_z^2\sigma_z^2 + 2w_x w_y Cov(r_x, r_y) + \\ 2w_x w_z Cov(r_x, r_z) + 2w_y w_z Cov(r_y, r_z)$$

Diversification and Risk (cont.)

- Portfolio standard deviation as a function of investment proportions
  - Minimum-variance portfolio: show the effect of diversification
  - Portfolio opportunity set for different values of the correlation coefficient
  - The best trade-off among different choices is a matter of personal preference (investor risk-aversion)

Determining efficient portfolios

- Usually undertake optimisation process
  - Determine “optimum” (depends on objectives) portfolio for each level of return
    - Determines combinations of risky assets
  - Can define constraints (eg no short sales – can’t hold securities in negative amounts)
- Markowitz optimisation is an example
  - Markowitz was the pioneer in this field
  - Presented as part of Ph.D. thesis

Markowitz Portfolio Theory

- Derived the expected rate of return for a portfolio of assets and an expected risk measure (variance)

- Assumptions (regarding investor behaviour)

  - Investors consider each investment alternative as being represented by a probability distribution of expected returns over some holding period.

  - Investors maximize one-period expected utility, and their utility curves demonstrate diminishing marginal utility of wealth.

  - Investors estimate the risk of the portfolio on the basis of the variability of expected returns.

  - Investors base decisions solely on expected return and risk, so their utility curves are a function of expected return and the expected variance of returns only.

Markowitz Portfolio Theory (cont.)

- For a given risk level, investors prefer higher returns to lower returns. Similarly, for a given level of expected return, investors prefer less risk to more risk.

Under this assumptions, ***a single asset or portfolio of assets is considered to be efficient if no other asset or portfolio of assets offers higher expected return with the same (or lower) risk, or lower risk with the same (or higher) expected return***

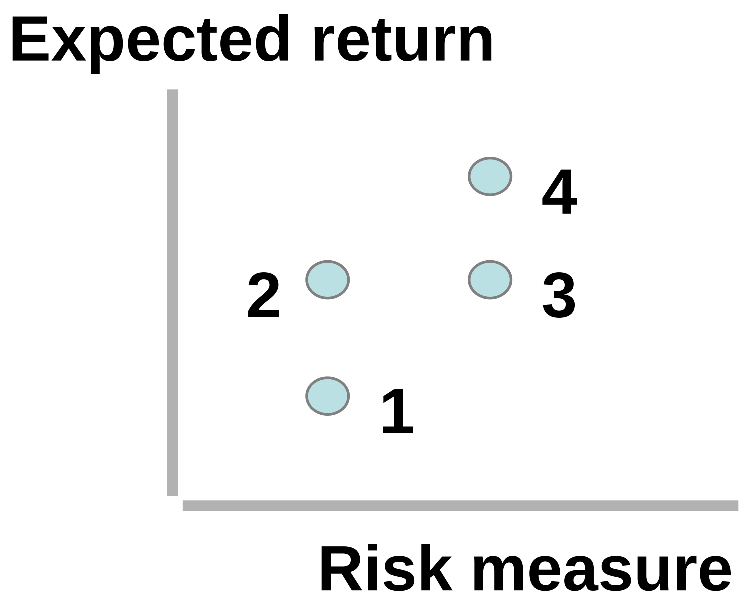
Markowitz process

- Develop forecasts of return and risk
  – Forward looking
  – May not be related to historical statistics

- Produce the efficient frontier of risky assets

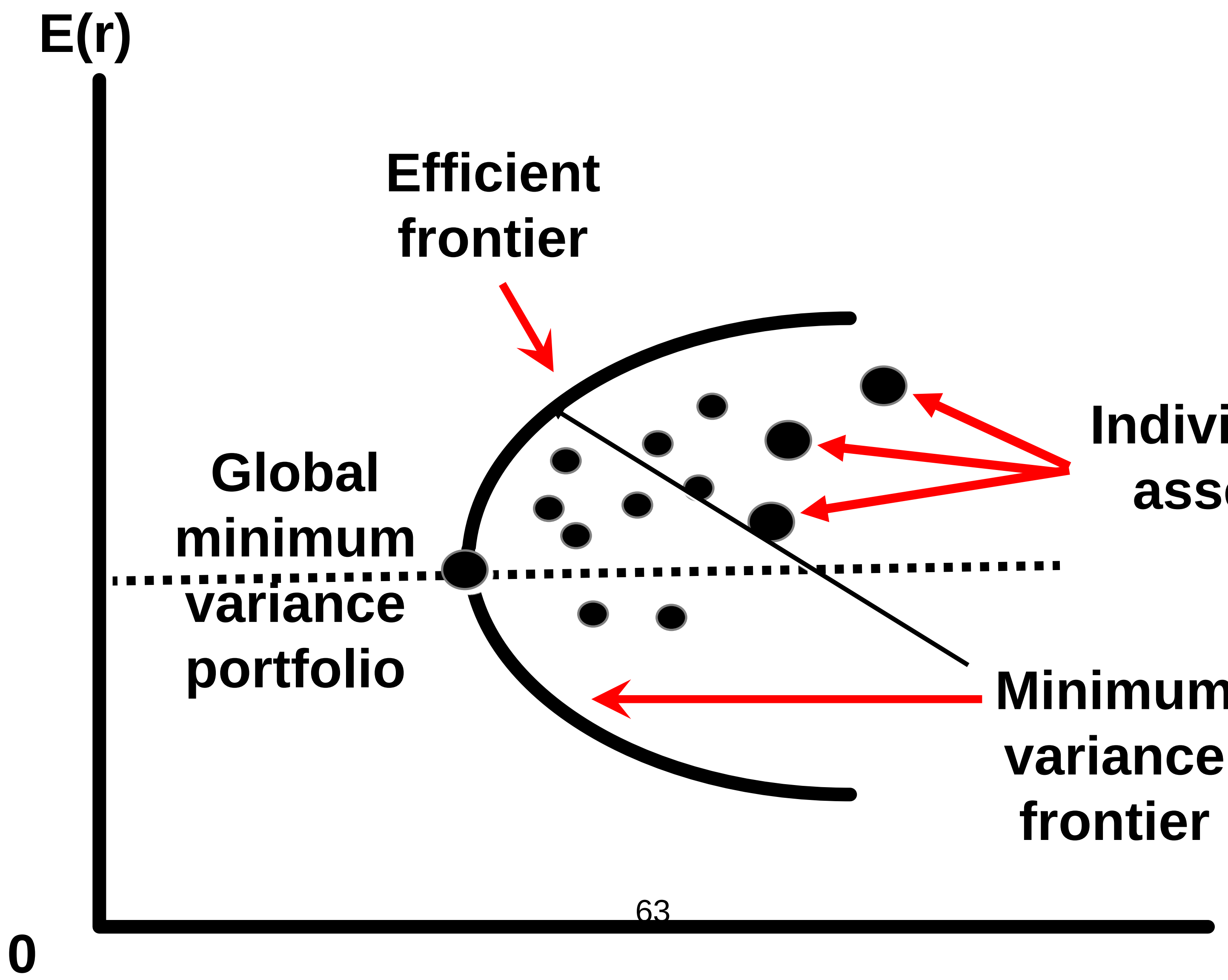
- Select the optimum risky portfolio
  – Based on risk-free rate
  – Maximises the reward-to-variability ratio

Dominance principle



2 dominates 1–higher return, same risk
2 dominates 3–lower risk, same return
4 dominates 3–higher return, same risk

Minimum-variance case: frontier of risky assets



Markowitz optimisation

$$\text{Choose} \quad w_i,\ w_j \ \text{ so as to :}$$

$$\min_{w_i, w_j} \sigma_P^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

$$\text{subject to :}$$

$$E(r_p^*) = \sum_{i=1}^{n} w_i E(r_i); \ \sum_{i=1}^{n} w_i = 1.00$$

Adding the Risk Free Asset

- Optimal combinations of assets
  - Represent linear combinations of two positions
  - Combine risk-free asset with optimal risky portfolio

- Given *rf*
  - A single risky portfolio will dominate
  - Its associated complete portfolio will dominate all other mixed portfolios

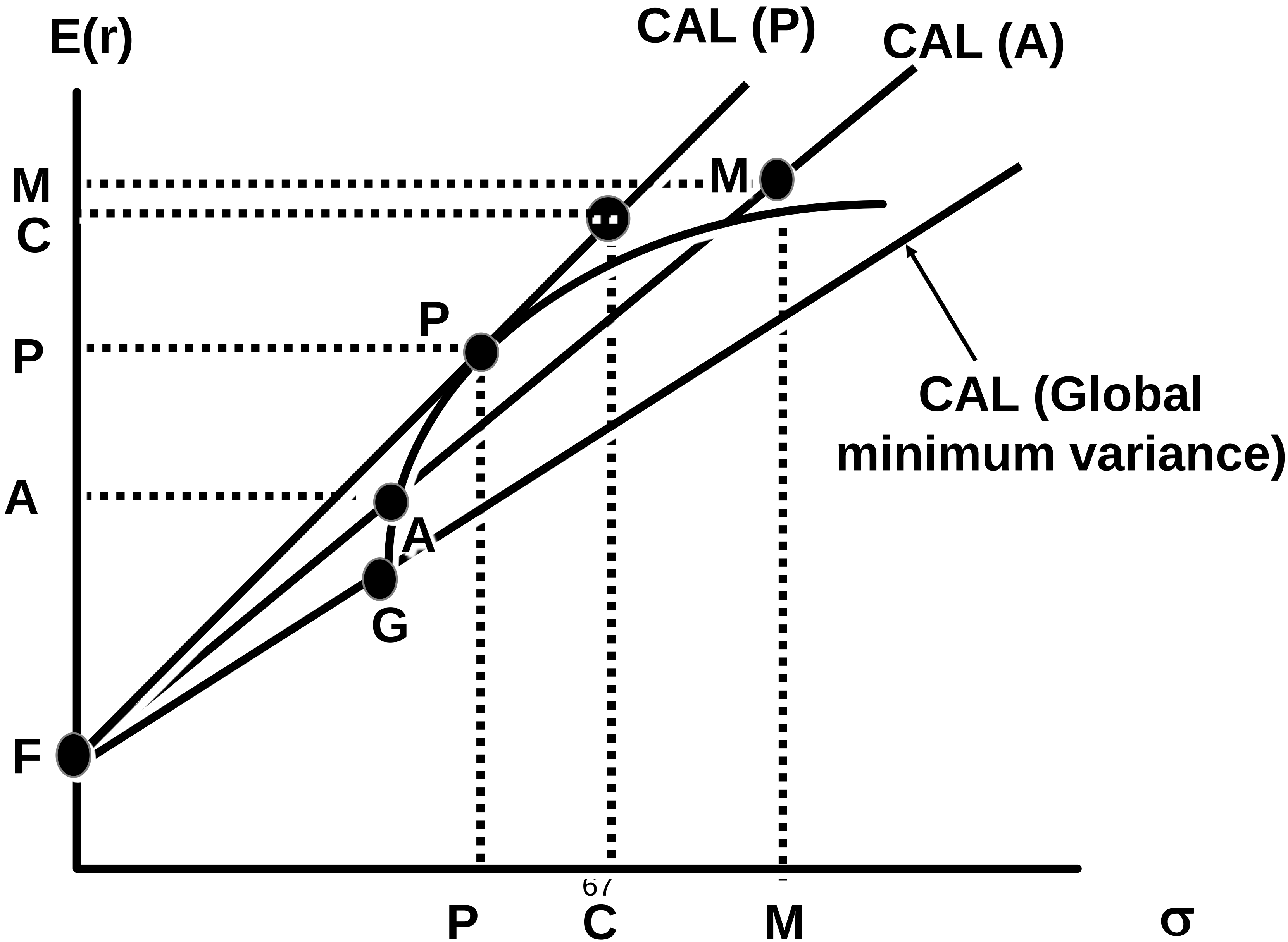
The Dominant CAL

- CAL(P) has the optimum risk-return trade-off

- Properties of CAL(P)
  - Portfolios combining P and F dominate other portfolios
  - Dominance is independent of risk preferences

$$\text{Slope} = \frac{E(r_P) - r_f}{\sigma} \text{ is maximised}$$

Possible CALs



An example using Markowitz

Calculation of average returns for optimisation

$$\overline{r_{ia}} = \frac{1}{T}\sum_{t=1}^{T} r_{it}$$

Use arithmetic mean return

Distribution of returns used in Markowitz

$$\sigma^2 = \frac{1}{T}\sum_{t=1}^{T}\left(r_{it} - \overline{r_i}\right)^2$$

Variance

An Example using Markowitz

A. Annualized Standard Deviation, Average Return, and Correlation Coefficients of International Stocks, 1980-1993

	St. Dev. (%)	Average Ret. (%)
US	21.1	15.7
Germany	25.0	21.7
UK	23.5	18.3
Japan	26.6	17.3
Australia	27.6	14.8
Canada	23.4	10.5
France	26.6	17.2

Correlation Matrix

	US	**Germany**	**UK**	**Japan**	**Australia**	**Canada**	**France**
US	**1.00**	0.37	0.53	0.26	0.43	0.73	0.44
Germany	0.37	**1.00**	0.47	0.36	0.29	0.36	0.63
UK	0.53	0.47	**1.00**	0.43	0.50	0.54	0.51
Japan	0.26	0.36	0.43	**1.00**	0.26	0.29	0.42
Australia	0.43	0.29	0.50	0.26	**1.00**	0.56	0.34
Canada	0.73	0.36	0.54	0.29	0.56	**1.00**	0.39
France	0.44	0.63	0.51	0.42	0.34	0.39	**1.00**

An Example using Markowitz

B. Covariance Matrix

	US	Germany	UK	Japan	Australia	Canada	France
US	445.21	195.18	262.80	145.93	250.41	360.43	246.95
Germany	195.18	625.00	276.13	239.40	200.10	210.60	418.95
UK	262.80	276.13	552.25	268.79	324.30	296.95	318.80
Japan	145.93	239.40	268.79	707.56	190.88	180.51	297.18
Australia	250.41	200.10	324.30	190.88	761.76	361.67	249.61
Canada	360.43	210.60	296.95	180.51	361.67	547.56	242.75
France	246.95	418.95	318.80	297.18	249.61	242.75	707.56

An Example using Markowitz

C. Border-Multiplied Covariance Matrix for the Equally Weighted Portfolio and Portfolio Variance

Portfolio weights	**US 0.1429**	**Germany 0.1429**	**UK 0.1429**	**Japan 0.1429**	**Australia 0.1429**	**Canada 0.1429**	**France 0.1429**
0.1429	9.09	3.98	5.36	2.98	5.11	7.36	5.04
0.1429	3.98	12.76	5.64	4.89	4.08	4.30	8.55
0.1429	5.36	5.64	11.27	5.49	6.62	6.06	6.51
0.1429	2.98	4.89	5.49	14.44	3.90	3.68	6.06
0.1429	5.11	4.08	6.62	3.90	15.55	7.38	5.09
0.1429	7.36	4.30	6.06	3.68	7.38	11.17	4.95
0.1429	5.04	8.55	6.51	6.06	5.09	4.95	14.44
1.0000	38.92	44.19	46.94	41.43	47.73	44.91	50.65
Portfolio variance	314.77						
Portfolio SD	17.7						
Portfolio Mean	16.5						

An Example using Markowitz

D. Border-Multiplied Covariance Matrix for the Efficient fronteir Portfolio with Mean of 16.5%
(after change of weights by solver)

Portfolio weights	**US 0.3467**	**Germany 0.1606**	**UK 0.0520**	**Japan 0.2083**	**Australia 0.1105**	**Canada 0.1068**	**France 0.0150**
0.3467	53.53	10.87	4.74	10.54	9.59	13.35	1.29
0.1606	10.87	16.12	2.31	8.01	3.55	3.61	1.01
0.0520	4.74	2.31	1.49	2.91	1.86	1.65	0.25
0.2083	10.54	8.01	2.91	30.71	4.39	4.02	0.93
0.1105	9.59	3.55	1.86	4.39	9.30	4.27	0.41
0.1068	13.35	3.61	1.65	4.02	4.27	6.25	0.39
0.0150	1.29	1.01	0.25	0.93	0.41	0.39	0.16
1.0000	103.91	45.49	15.21	61.51	33.38	33.53	4.44
Portfolio variance	297.46						
Portfolio SD	17.2						
Portfolio Mean	16.5						

Efficient Frontier with seven countries