FINA 1082 Financial Management

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Lecture 2

The arbitrage-free Approach to Bond Valuation

Theories of Term Structure of Interest Rates

The measurement of Interest Rate Risk

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The arbitrage-free Approach to Bond Valuation

The traditional valuation approach is deficient because it uses a single discount rate (the appropriate YTM) to find the present value of the future cash flows with no regard given to the timing of those cash flows.

Cash flows received in year 1 on a 20 year bond are discounted at the same rate as the cash flows received in 20 years!

Arbitrage Free Valuation Model

$$P = \frac{CF_1}{(1+r_1)} + \frac{CF_2}{(1+r_2)^2} + \frac{CF_3}{(1+r_3)^3} + \ldots + \frac{CF_n}{(1+r_n)^n}$$

This model treats each separate cash flow paid by a fixed-income security as if it were a stand-alone zero-coupon bond. These discount rates are called spot rates.

Example

Give the following Treasury spot rates, calculate the arbitrage-free value of a 5% coupon, 2 year treasury note.

Maturity	Spot Rate
0.5 years	4.0%
1.0	4.4%
1.5	5.0%
2.0	5.2%

The arbitrage-free price of the note is:

$$P = \frac{CF_1}{(1+r_1)} + \frac{CF_2}{(1+r_2)^2} + \frac{CF_3}{(1+r_3)^3} + \ldots + \frac{CF_n}{(1+r_n)^n}$$

$$P = \frac{\$2.50}{(1.020)} + \frac{\$2.5}{(1.022)^2} + \frac{\$2.5}{(1.025)^3} + \frac{\$102.5}{(1.026)^4} = \$99.66 \text{ per } \$100 \text{ of par value}$$

Profiting from Arbitrage Opportunities: Stripping and Reconstituting Bonds

Stripping

Suppose the same 2-year, 5% coupon Treasury note is priced at $95.00, which is below its arbitrage-free value of $99.66. What action should arbitrageurs take and what will be the affect of their actions?

Because the note is priced below its arbitrage-free value, its zero-coupon cash flow “pieces ”are worth more than the note it self. Therefore, an arbitrage profit could be earned by:

- Buying the undervalued note at $95.00

- Stripping the note of its individual cash flows

- Selling the individual cash flows “pieces” as zero-coupon bonds for $99.66 and earning arbitrage profit of $4.66 per $100 of investment.

As this arbitrage is performed:

• The increased demand for the notes will cause their prices to increase and their yields maturity to fall

• The increased supply of zero-coupon bond “pieces” will cause the prices of zero-coupon bonds to fall and their yields (spot rates) to rise.

• These forces will quickly eliminate the arbitrage opportunity

Reconstituting

The 2-year, 5% coupon Treasury note is priced at $100. Its arbitrage-free value is $99.66. What action should arbitrageurs take and what will be the effect of their actions?

Because the note is priced above its arbitrage-free value, it is overpriced relative to the value of its zero-coupon cash flow “pieces”. Therefore, an arbitrage profit can be earned by:

- Buying the zero coupon “pieces” in the zero-coupon treasury market for $99.66

- Reconstituting the note from these zero-coupon Treasuries.

- Selling the reconstituted note for $100 to earn an arbitrage profit of $0.34 for every $99.66 of original investment.

As dealers perform this arbitrage:

- The increased demand for Treasury zero-coupon bonds will drive their prices up and their yields (spot rates) down.

- The increased supply of reconstituted 2-year, 55 coupon treasuries will drive their prices down and their yields-to-maturity up

- These forces will quickly eliminate the arbitrage opportunity

Arbitrage Example

We observe two types of bonds: T-bills and coupon bonds. A one-year T-bills pays 1000 in one year, a two year T-bill pay 1000 in two years and a three year T-bill pays 1000 in three years.

There are no coupon interests on T-bills. The coupon bond is a 5% three-year bond with a face value of 1000. Thus the cash flow from the coupon bond are: In the first year, you receive 50, in the second year 50, and in the last year 1050. We observe the following prices:

Type of Bond	Price	Yield
One year T-bill	943.4	$\frac{1000}{1+r} = 943{,}4 \leftrightarrow r = 6\%$
Two year T-bill	873.44	$\frac{1000}{(1+r)^2} = 873{,}44 \leftrightarrow r = 7\%$
Three year T-bill	793.83	$\frac{1000}{(1+r)^3} = 793{,}83 \leftrightarrow r = 8\%$
Coupon bond	1.000	

Assumption: We can borrow funds at the above rates and short sell the securities without any costs

- No arbitrage profit:
  - Using no wealth
  - No risk
  - Positive return
- The first condition requires a long and short position
- To satisfy the second condition (no risk) we need to match the cash flows from the long and short positions

- Short sell (borrow) 20 coupon bonds and buy 1 one year T-bill, 1 two year T-bill and 21 three year T-bills $\rightarrow$ we do not use any of our own wealth.

Pricing by arbitrage

Cash flows from bond transactions

	Number of Bonds	Price	Cash flows at time: 0	1	2	3
Coupon Bonds	20	1,000	20,000	-1,000	-1,000	-21,000
Short position - Loan						**Long Position**
One year T-bill	1	943,4	-943,4	1,000	0	0
Two year T-bill	1	873,44	-873,44	0	1,000	0
Three year T-bill	21	793,83	-16,670.43	0	0	21,000
TOTAL	3		1,512.73	0	0	0

We have an arbitrage profit of 1,512.73, with no risk and using none of our own funds

What will happen:

- Investors will sell the coupon bonds → prices start to drop
- Investors will buy T-bills → price start to increase
- The price of the coupon bond that is consistent with the no *arbitrage condition* is

924.3635

Pricing using discounted cash flow

$$P = \frac{50}{1+0.06} + \frac{50}{(1+0.07)^2} + \frac{1050}{(1+0.08)^3} = 47.17 + 43.67 + 833.52$$

924.37

- Each period has different discount rates since the opportunity costs for each cash flow (coupon payment) is different, i.e. different T-bills
- Do not look for arbitrage profits in markets where is simple to construct arbitrage portfolios
  - Bond market
  - Option market
  - Forward markets for exchange rates

Theories of Term Structure of Interest Rates

What is the information in the yield curve?

How can it be explained and interpreted changes in the yield curve?

Three main theories:

1) The pure expectation theory (unbiased expectations theory)

2) The liquidity preferences theory (or liquidity premium theory)

3) The market segmentation theory

Pure Expectations Theory

Makes simple link between the yield curve and investors' expectations about future interest rates.

Also, because long-term interest rates are possible linked to investors expectations about future inflation, it also address economic interpretations.

Explains the term structure in terms of expected future short-term interest rates.

The market will sets the yield on a two year-bond so that the return on a two-year bond is approximately equal to the return on a one-year bond plus the expected return on a one-year bond purchased one year from today.

Therefore:
Raising term structure indicates that the market expects short-term rates to raise in the future

Shape of term structure	Implications according to pure expectations theory
Upward sloping (normal)	Rates expected to rise
Downward sloping (inverted)	Rates expected to decline
Flat	Rates not expected to change

Under the hypothesis that interest rates reflect the sum of a relatively stable real rate of interest plus a premium for expected inflation:

If short-term rates are expected to rise, investors expect inflation to rise as well.

Shortcomings: assumes that investors are indifferent to interest rate risk and any other factors associated with investing in bonds with different maturities.

Liquidity Preference Theory

Market participants want to be compensated for the interest rate risk associated with holding longer-term bonds.

Therefore, the term structure of interest rates is determined by:

1) Expectations about future interest rates
2) Yield premium for interest rate risk (more interest rate risk, the less the liquidity)

Since interest rate risk increases with maturity, yield premium increases with maturity

Shape of term structure	Implications according to Liquidity Preference Theory
Upward sloping (normal)	Rates expected to rise, or will be unchanged or even fall (but with yield premium increasing with maturity fast enough to produce an upward sloping of yield curve)
Downward sloping (inverted) or flat	Rates expected to fall, given the theory's prediction that the yield premium for interest rate risk increases with maturity

Market Segmentation Theory

Each maturity "sector" is an independent or segmented market for purposes of determining the interest rate in the maturity "sector".

Two major groups of investors:

1) Those who manage funds versus a broad-based bond market index, and
2) those that manage funds against liabilities.

The 2nd group will restrict their activities to the maturity sector that provides the best match with the maturity of their liabilities (basic principle of asset-liability management).

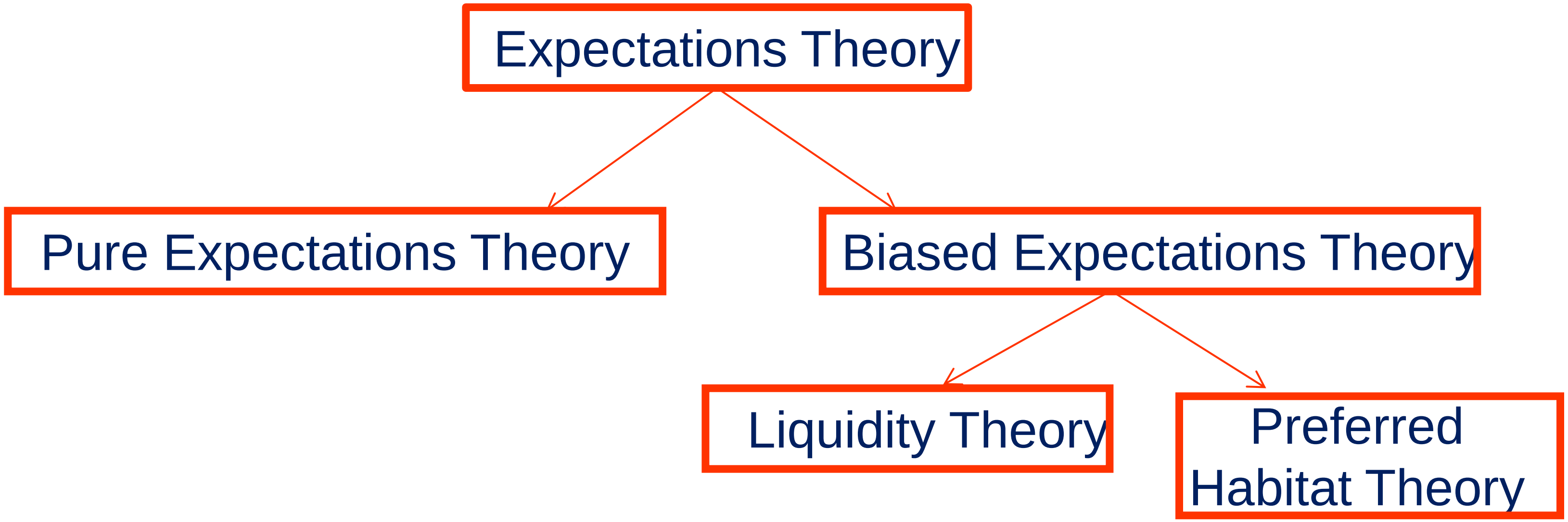
Defined benefit pension fund: investment in long-term maturity sector of the bond market

Commercial banks: will focus in short-term fixed income investments (since their liabilities are mainly short-term).

Preferred habitat theory: variant of the market segmentation theory

Investors might be willing to shift out for their preferred maturity sector if a incentive yield premium exist.

Implication: Under the preferred habitat theory any shape of the yield curve is possible.



The measurement of Interest Rate Risk

Most obvious way: Re-value the bond when interest rates change

Example:

$10 million par value position in a 9% coupon 20-year bond (option free).

Current price: 134.6722 for a YTM of 6%. Market value of the position is $13,467,220

Three scenarios:

1) 50 basis point increase
2) 100 basis point increase
3) 200 basis point increase

Scenario	Yield change (bp)	New yield	New Price	New Market Value ($)	Percentage Change in Market Value (%)
1	50	6.5%	127.7606	12,776,050	-5.13%
2	100	7.0%	121.3551	12,135,510	-9.89%
3	200	8.0%	109.8964	10,989,640	-18.40%

Price Volatility Characteristics of Bonds

Price of bond changes in the opposite direction to a change in the bond's yield.

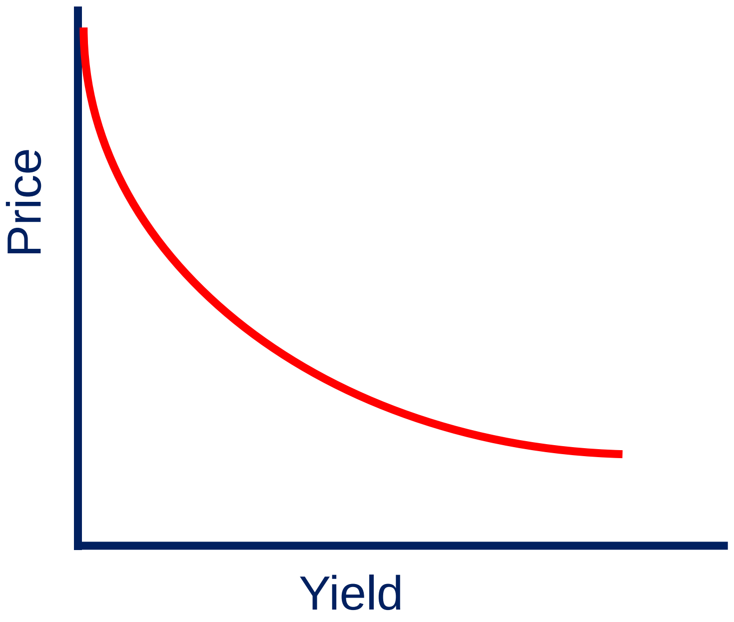
The percentage price change is not the same to all bonds

	Price ($)			
Yield (%)	6%/ 5 year	6%/20 year	9%/5 year	9%/20 year
4.00	108.9826	127.3555	122.4565	168.3887
5.00	104.3760	112.5514	117.5041	150.2056
5.50	102.1600	106.0195	115.1201	142.1367
5.90	100.4276	101.1157	113.2556	136.1193
5.99	100.0427	100.1157	112.8412	134.8159
6.00	100.0000	100.0000	112.7953	134.6722
6.01	99.9574	99.8845	112.7494	134.5287
6.10	99.5746	98.8535	112.3373	133.2472
6.50	97.8944	94.4479	110.5280	127.7605
7.00	95.8417	89.3225	108.3166	121.3551
8.00	91.8891	80.2072	104.0554	109.8964

Instantaneous Percentage Price Change for Four Hypothetical Bonds (Initial yield for all four bonds is 6.%)

	Percentage Price Change			
Yield (%)	6%/ 5 year	6%/20 year	9%/5 year	9%/20 year
4.00	8.98	27.36	8.57	25.04
5.00	4.38	12.55	4.17	11.53
5.50	2.16	6.02	2.06	5.54
5.90	0.43	1.17	0.41	1.07
5.99	0.04	0.12	0.04	0.11
6.01	-0.04	-0.12	-0.04	-0.11
6.10	-0.43	-1.15	-0.41	-1.06
6.50	-2.11	-5.55	-2.01	-5.13
7.00	-4.16	-10.68	-3.97	-9.89
8.00	-8.11	-19.79	-7.75	-18.40

Price/yield relationship for a hypothetical option-free bond



1) Although the price moves in the opposite direction from the change in the yield, percentage change is not the same for all bonds
2) For small changes in the yield, the percentage price changes for a given bond is roughly the same, whether the yield increases or decreases.
3) For large changes in yield, the percentage price change is not the same for an increase in yield as it is for a decrease in yield
4) For a given large change in yield, the percentage price increase is greater than the percentage price decrease.

Duration/Convexity approach

Duration is a measure of the approximate price sensitivity of a bond to interest rate changes. It is the approximate percentage change in price for a 100 basis points change in rate.

Duration: average lifetime of a debt security's stream of payments

Two bonds with the same term to maturity does not mean that they have the same interest-rate risk

Macaulay Duration

Is the weighted average time-to-maturity of the cash flows of a bond. In the Macaulay, and all other duration measures, the weighting of the cash flows is based on their discounted present value, rather then their nominal value.

$$Macaulay\ Duration = \frac{\sum (n)(PV\ of\ Cash\ Flows)}{k \times Bond\ Price}$$

Where:
n is the number of periods until each cash flow is paid
k is the number of times coupon interest is paid per year

Calculating Macaulay Duration on a $1,000 ten-year 10% Coupon Bond when its Interest Rate is 10%

(1)	(2)	(3)	(4)	(5)
Year	Cash Payments (Zero-Coupon Bonds)	Present Value of Cash Payments	Weights (% of total)	Weighted Maturity (1×4)/100
1	**100**	**90.91**	**9.091**	**0.09091**
2	**100**	**82.64**	**8.264**	**0.16528**
3	**100**	**75.13**	**7.513**	**0.22539**
4	**100**	**68.30**	**6.830**	**0.27320**
5	**100**	**62.09**	**6.209**	**0.31045**
6	**100**	**56.44**	**5.644**	**0.33864**
7	**100**	**51.32**	**5.132**	**0.35924**
8	**100**	**46.65**	**4.665**	**0.37320**
9	**100**	**42.41**	**4.241**	**0.38169**
10	**100**	**38.55**	**3.855**	**0.38550**
10	**1000**	**385.54**	**38.554**	**3.85500**
Total		1,000.00	100.00	**6.75850 (years)**

Effective Duration

If the yield increased by a small amount Δr, from r_0 to r_+, the price of the bond will decrease from P_0 to P_-.

A bond's effective duration measures how sensitive the return on the bond (measured as the percentage change in its price) will be to the change in interest rates.

$$DE = \frac{Percentage\ Change\ in\ Price\ of\ Bond}{Change\ in\ Interest\ Rates}$$

$$DE = \frac{\Delta P/P}{\Delta rates} = \frac{(P_- - P_+)/P_0}{2\Delta r}$$

$$DE = \frac{P_- - P_+}{2P_0\Delta r}$$

Example:

A 7% coupon, 5-year bond, yielding 6% is priced at 104.265. If its yield declines by 25 basis points to 5.75%, the bond's price will increase to 105.366. On the other hand, if its yield increases by 25 basis points to 6.25%, the price of the bond will decline to 103.179. Compute the effective duration of the bond under these conditions.

$$DE=\frac{P_{-}-P_{+}}{2P_0\Delta r}=\frac{105.366-103.179}{2(104.265)(.0025)}=4.2$$

If this bond's yield increases by 1% (apparently because interest rates have increased by 100 basis points), the price of the bond will fall by approximately 4.2%.

Dollar Duration

Measures the dollar market value change resulting from a 100 basis point change in yield:

$$\text{Dollar duration} = -D_E \times (\$\ \text{Market Value}) \times \Delta r$$

Example:

A manager has a holding of XYZ bond with a current market value of \$25 million and a duration of 5.4. If the bond's yield dropped by 100 basis points, what would be the change in the market value.

$$\text{Dollar duration} = -D_E \times (\$\text{ Market Value}) \times \Delta r$$
$$= -5.4 \times (\$25{,}000{,}000) \times -0.0100$$
$$= +\$1{,}350{,}000$$

Application of Effective Duration

From the basic formula: percentage change in the price of bond will approximately equal its effective duration times any change that occurs in its yield, but in opposite direction

$$\%\text{ Change in Price of Bond} = \frac{\Delta P_B}{P_B} = -D_E \times \Delta r$$

Example:

A 7% coupon, 5-year bond, priced at 104.265 with duration of 4.2 has a YTM of 6%. Estimate the percentage change in the price and the new price of the bond if its YTM declines by 50 basis points from 6% to 5.5%.

$$\%\text{ Change in Price of Bond} = \frac{\Delta P_B}{P_B} = -D_E \times \Delta r = -4.2(-.005) = 2.1\%$$

$$P_{NEW} = P_0\left(1 + \frac{\Delta P_B}{P_B}\right) = 104.265 \times (1.021) = 106.455$$

Modified Duration

Is an adjusted measure of the Macaulay duration that produces a more accurate estimate of how much the percentage change in the price of a bond will be per 100 basis points change in the interest rate.

$$\text{Modified Duration (D*)} = \frac{\text{Macaulay Duration}}{(1 + \text{yield} / k)}$$

Where

Yield is the yield-to-maturity of the bond

K is the number of periodic payment (compounding) periods per year

Interpretations of Duration

Three interpretations:

1. Effective duration is the first derivative of the price-yield relationship of a security.

$$\mathrm{DE} = -\frac{dP/dr}{\mathrm{P}}$$

While correct, this interpretation only has meaning to the mathematically inclined.

2. Unadjusted duration is a weighted average of time. Indeed Macaulay's (unadjusted) duration is measured in units of time (years) .Thus a Macaulay's duration may be said to be 8 years. This may be meaningful to an investment professional who understands that a bond with a duration of 8 years is more volatile than a bond with a duration of 3 years.

3. Effective duration is a measure of how sensitive the return on a bond is to small changes in interest rates

$$D_E = - \frac{\Delta P / P}{\Delta r}$$

Indicates that if a bond has a duration of 3, its price will rise or fall 3% every time interest rates fall or rise by 100 basis points. A bond of duration of 6, has therefore twice as much interest risk as the first.

However, that effective duration only measures interest rate risk – it does not measure any other type of risk (credit risk, currency risk, liquidity risk, and so forth).

Calculating Bond Portfolio Duration

$$D_{Portfolio} = W_1 D_1 + W_2 D_2 + ... + W_n D_n$$

Example:

A barbell portfolio is constructed with 60% of its value invested in a 4-year bond with an effective duration of 3.0 and 40% of its value invested in a 15-year bond with an effective duration of 10.0. What is the effective duration of the portfolio?

$$D_E = W_1 D_{E1} + W_2 D_{E2} = .6(3) + .4(0) = 5.8$$

Bond Convexity

Convexity is a measure of the curvature or 2nd derivative of how the price of a bond varies with interest rate, i.e. how the duration of a bond changes as the interest rate changes.

Higher the convexity, the more sensitive the bond price is to the change in interest rates.

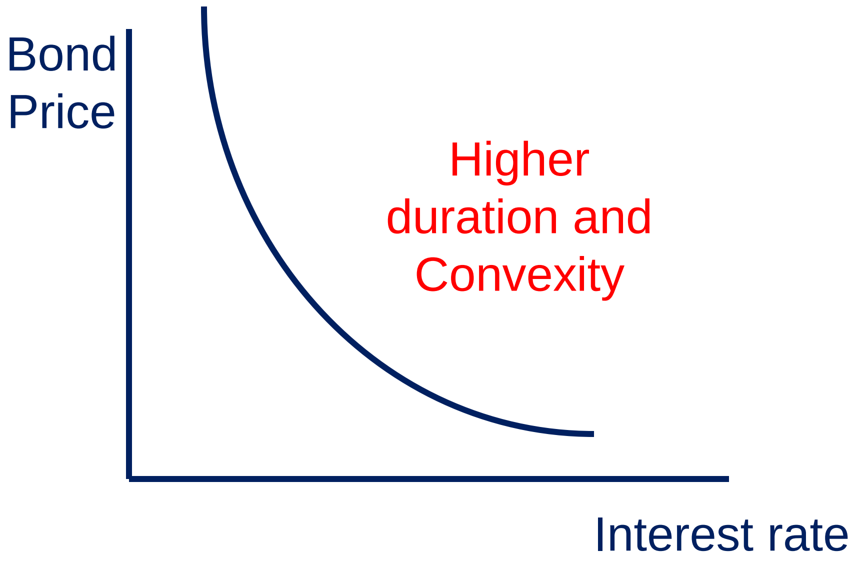
Duration: linear measure of 1st derivative of how the price of a bond changes in response to interest rate changes.

- As interest rates change, the price is not likely to change linearly, but instead it would change over some curved function of interest rates.
- The more curved the price function of the bond is, the more inaccurate duration is as a measure of the interest rate sensitivity.

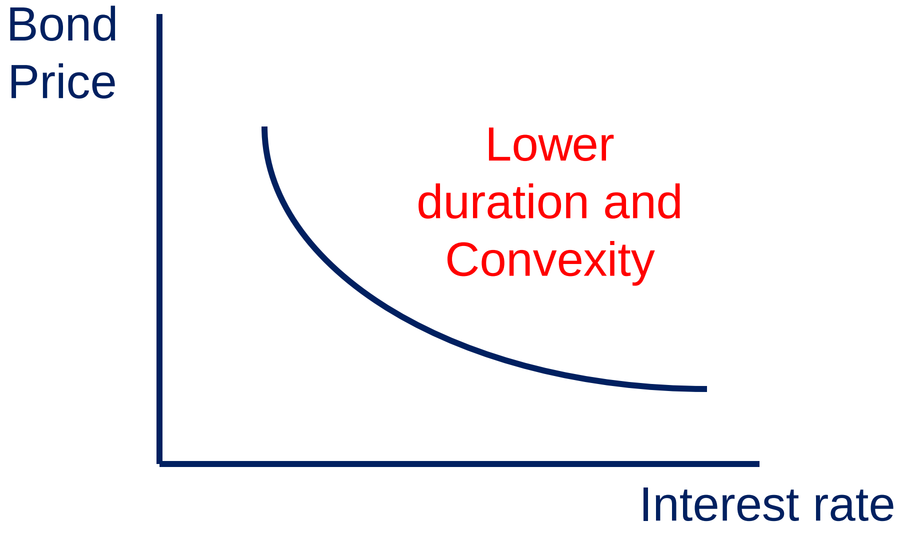
Why bond convexities differ?

The price sensitivity to parallel changes in the term structure of interest rates is highest with a zero-coupon bond and lowest with an amortizing bond.

30-year maturity,
8% coupon
Straight bond



5-year maturity,
8% coupon
Straight bond



The divergence from the straight line is called convexity

As interest rates decline duration projects the bond's price to rise, but the actual increase in price is greater than duration projects.

If interest rates rise, the duration line projects the bond's price will fall, but the actual decline in price is less than duration projects.

Positive convexity: a larger increase in price than decrease in price, for the same change in rates. The upside is greater than downside.

All bonds that are free of embedded options will have positive convexity, but the actual degree of duration and convexity of bonds will vary.

Observations about Straight Bonds

Generalizations about option-free bonds:

1) The price-yield relationship is negative, meaning that the price and yield move in opposite directions.

2) For small changes in interest rates, the upside and downside price moves are approximately equal in size.

3) For large changes in interest rates, the upside and downside will diverge from the duration life.

4) For option-free bonds, the upside will be greater than the downside price move for the same change in rates.

5) The actual level of duration and convexity will vary with the terms of the bond.

6) As interest rates change from high to low, the duration of the bond will increase (and so will be convexity).

Calculating Convexity

$$CE = \frac{P_- + P_+ - 2P_0}{2P_0 \Delta r^2}$$

Where:

P_0 is the initial price of the bond

P_- is the price if the yields decline by Δr

P_+ is the price if the yields increases by Δr

Δr is the change in the bond's yield, which can also be viewed as being a change in interest rates if the bond's yield spread is assumed to be constant

Bond Sensitivity to Interest Rate Changes

Duration and convexity can be used to project how the price of a bond, or the value of a portfolio of bonds, will change for a given change in interest rates

$$\%\Delta P = \frac{\Delta P}{P} = -D_E \Delta r + C_E \Delta r^2$$

Where:

P_0 is the initial price of the bond

ΔP is the change in the price of the bind associated with Δr

D_E is the effective duration of the bond

C_E is the effective convexity of the bond

Δr is the change in the bond’s yield, which can also be viewed as being a change in interest rates if the bond’s yield spread is assumed to be constant

Note: duration and convexity are simply variations on the first and second derivative of the price function of the bond.

Example:

A 10-year, 8% coupon bind is selling at 93.5000, with a yield-to-maturity of 9.0%. A bond valuation model indicates that if the yield on the bond is increased by 50 basis points to 9.5%, its price will fall to 90.4520, and if the bond's yield is shocked downward by 50 basis points to 8.5%, the bond's value will increase to 96.6764.

What is the effective duration and effective convexity of this bond at the current price of 93.50000? If interest rates decreased by 100 basis points, what is the estimated percentage change in the price of the bond?

$$D_E = \frac{P_- - P_+}{2P_0 \Delta r^2} = \frac{96.6764 - 90.4520}{2(93.5000)(0.0050)} = 6.657$$

$$C_E = \frac{P_- + P_+ - 2P_0}{2P_0 \Delta r^2} = \frac{96.6764 + 90.4520 - 2(93.5000)}{2(93.5000)(0.0050)^2} = 27.465$$

$$\%\Delta P = \frac{\Delta P}{P} = -D_E \Delta r + C_E \Delta r^2 = -6.657(-0.01) + 27.465(0.01)^2 = 6.93\%$$

Magnitude of changes in interest rates

If Δr is small, Δr^2 will approach zero.

The convexity adjustment $C_E\Delta r^2$ will be insignificant

Convexity is not important in estimating the price change of a bond for small changes in yields.

Example

Bond with and effective duration of 3.94 and a convexity of 9.685

For a large change in rates convexity matters:

If interest rates rise by 1% (100 basis points)

$$\%\Delta P = \frac{\Delta P}{P} = -D_E\Delta r + C_E\Delta r^2 = -3.940(0.010) + 9.685(0.010)^2 = -3.84\%$$

If interest rates decrease by 1% (100 basis points)

$$\%\Delta P = \frac{\Delta P}{P} = -D_E\Delta r + C_E\Delta r^2 = -3.940(0.010) + 9.685(-0.010)^2 = 4.04\%$$

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For a small change in rates convexity does not matter:

If interest rates increase by 1 basis point (0.0001)

$$\%\Delta P = \frac{\Delta P}{P} = -D_E \Delta r + C_E \Delta r^2 = -3.940(0.0001) + 9.685(0.0001)^2 = -0.04\%$$

If interest rates decrease by 1 basis point (0.0001)

$$\%\Delta P = \frac{\Delta P}{P} = -D_E \Delta r + C_E \Delta r^2 = -3.940(-0.0001) + 9.685(-0.0001)^2 = +0.04\%$$

$C_E \Delta r^2$ is virtually zero (at four decimal places), so the upside and downside percentage price movements are virtually equal.