Fixed Income Investment

Session 3
April, 25th, 2013
(morning)

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Lecture 3

Binomial Interest Rate Trees and the Valuation of Bonds with Embedded Options

- A 2 and 3-Period Option-Free Bond
- A 2 and 3-Period Callable Bond
- Other Embedded Option Features
  - Valuing a 3-Period Putable Bond
  - Valuing Sinking-Fund Bonds
  - Valuing Convertible Bonds
  - Valuing Callable Convertible Bonds

Valuation of Bonds with Embedded Options

The inclusion of option features in a bond contract makes the evaluation of such bonds more difficult.

A 10-year, 10% callable bond issued when interest rate are relatively high may be more like a 3-year bond, given that a likely interest rate decrease would lead the issuer to buy the bond back.

Determining the value of such a bond requires taking into account not only the value of the bond's cash flow, but also the value of the call option embedded in the bond.

One way to capture the impact of a bond's option feature on its value is to construct a model that incorporates the *random paths* that interest rates follow over time.

One such model is the *binomial interest rate tree*.

Patterned after the binomial option pricing model, this model assumes that interest rates follow a *binomial process* in which in each period the rate is either *higher or lower*

Model

Assume a one-period, risk-free spot rate (S) follows a process in which in each period the rate is equal to a proportion u times its beginning-of-the-period value or a proportion d times its initial value, where u is greater than d.

After one period, there would be two possible one-period spot rates:

$$Su = uS_0 \text{ and } Sd = dS_0.$$

If the proportions u and d are constant over different periods, then after two periods there would be three possible rates:

$$S_{uu} = u^2S_0 \qquad S_{ud} = udS_0 \qquad S_{dd} = d^2S_0$$

$$S_0$$

$$S_u = uS_0$$

$$S_d = dS_0$$

$$S_{uu} = u^2S_0$$

$$S_{ud} = udS_0$$

$$S_{dd} = d^2S_0$$

$$S_{uuu} = u^3S_0$$

$$S_{uud} = u^2dS_0$$

$$S_{udd} = ud^2S_0$$

$$S_{ddd} = d^3S_0$$

Example:

Given
The current one-period spot rate: $S_0 = 10\%$
The upward parameter: $u = 1.1$
The downward parameter: $d = .95$.

The two possible one-period rates after one period are 11% and 9.5%

The three possible one-period rates after two periods are 12.1%, 10.45%, and 9.025%.

The four possible rates are three periods are 13.31%, 11.495%, 9.927%, and 8.574%.

$$S_0 = 10.00\%$$

$$S_u = uS_0 = 11.00\%$$

$$S_d = dS_0 = 9.5\%$$

$$S_{uu} = u^2S_0 = 12.10\%$$

$$S_{ud} = udS_0 = 10.45\%$$

$$S_{dd} = d^2S_0 = 9.025\%$$

$$S_{uuu} = u^3S_0 = 13.31\%$$

$$S_{uud} = u^2dS_0 = 11.495\%$$

$$S_{udd} = ud^2S_0 = 9.927\%$$

$$S_{ddd} = d^3S_0 = 8.574\%$$

Valuing a 2-Period Option-Free Bond

Given the possible one-period spot rates, suppose we want to value a bond with the following features:

- The bond matures in two periods
- The bond has no default risk
- The bond has no embedded option features – option-free bond
- The bond pays an 8% coupon each period
- The bond pays a $100 principal at maturity

Since there is no default or call risk, the only risk an investor assumes in buying this bond is market risk. This risk occurs at time period one.

At that time, the original two-period bond will have one period to maturity where there is a certain payoff of 108. We don't know, though, whether the one-period rate will be 11% or 9.5%:

If the rate is 11%, then the bond would be worth 97.297:

$$\boxed{B_u = 108/1.11 = 97.297}$$

If the rate is 9.5%, the bond would be worth 98.630:

$$\boxed{B_d = 108/1.095 = 98.630}$$

Given the two possible values in Period 1, the current value of the two-period bond can be found by calculating the present value of the bond's expected cash flows in Period 1.

If we assume that there is an equal probability (q) of the one-period spot rate being higher (q = 0.5) or lower (1 – q = 0.5), then the current value of the two-period bond (B_0) would be 96.330:

$$B_0 = \frac{q\left[B_u + C\right] + (1-q)\left[B_d + C\right]}{1+S_0}$$

$$B_0 = \frac{.5\left[97.297+8\right] + .5\left[98.630+8\right]}{1.10} = 96.330$$

$Coupon = 8,\ F = 100$

$S_0 = .10, u = 1.1,\ d = .95$

$S_0 = 10\%$

$$B_0 = \frac{.5(97.297 + 8) + .5(98.630 + 8)}{(1.10)}$$

$$= 96.330$$

$S_u = 11\%$

$$B_u = \frac{108}{(1.11)} = 97.297$$

$S_d = 9.5\%$

$$B_d = \frac{108}{(1.095)} = 98.630$$

$B_{uu} = 108$

$B_{ud} = 108$

$B_{dd} = 108$

Valuing a 2-Period Callable Bond

Suppose that the two-period, 8% bond has a call feature that allows the issuer to buy back the bond at a call price (CP) of 98.

Using the binomial tree approach, this call option can be incorporated into the valuation of the bond by determining at each node in Period 1 whether or not the issuer would exercise his right to call.

The issuer will find it profitable to exercise whenever the bond price is above the call price (assuming no transaction or holding costs).

In general, since the bond is only exercised when the call price is less than the bond value, the value of the callable bond in Period 1 is therefore the minimum of its call price or its binomial value:

$$\boxed{B_t^C = Min[B_t, CP]}$$

Rolling the two callable bond values in Period 1 of 97.297 and 98 to the present, we obtain a current price of 96.044:

$$\boxed{B_0^C = \frac{.5[97.297+8] + .5[98+8]}{1.10} = 96.044}$$

$$Coupon = 8,\ F = 100,\ CP = 98$$
$$S_0 = .10, u = 1.1,\ d = .95$$

$$S_0 = 10\%$$
$$B_0 = \frac{.5(97.297+8) + .5(98+8)}{(1.10)}$$
$$= 96.044$$

$$S_u = 11\%$$
$$B_u = \frac{108}{(1.11)} = 97.297$$
$$B_u^C = Min[B_u, CP]$$
$$B_u^C = Min[97.297, 98] = 97.297$$

$$S_d = 9.5\%$$
$$B_d = \frac{108}{(1.095)} = 98.630$$
$$B_d^C = Min[B_d, CP]$$
$$B_d^C = Min[98.630, 98] = 98$$

$$B_{uu} = 108$$

$$B_{ud} = 108$$

$$B_{dd} = 108$$

In our example:

When the one-period spot rate is 9.5% in Period 1 and the bond is priced at 98.630, it is profitable for the issuer to call the bond. The price of the bond in this case would be the call price of 98.

When the one-period spot rate is 11% in Period 1, it is not profitable for the issuer to exercise the call. The price of the bond in this case remains at 97.297

Note:
In the 9.5% case, the issuer could buy the bond back at 98 financed by issuing a one-year bond at 9.5% interest. One period later the issuer would owe 98(1.095) = 107.31; this represents a savings of 108-107.31 = 0.69. Note, the value of that savings in period one is .69/1.095 = 0.63, which is equal to the difference between the bond price and the call price:

$$98.630 - 98 = .63.$$

Note:

The bond's embedded call option lowers the value of the bond from 96.330 to 96.044.

At each of the nodes in Period 1, the value of the callable bond is determined by selecting the minimum of the binomial bond value or the call price, and then rolling the callable bond value to the current period.

Instead of using a price constraint at each node, the price of the callable bond can alternatively be found by determining the value of the call option at each node, V_t^C, and then subtracting that value from the noncallable bond value:

$$\boxed{B_t^C = B_t^{NC} - V_t^C}$$

In this two-period case, the values of the call option are equal to their intrinsic values, IV (or exercise values). The intrinsic value is the maximum of B_t^{NC} - CP or zero:

$$\boxed{V_t^C = Max[B_t^{NC} - CP, 0]}$$

In the example

When the one-period spot rate is 9.5% in Period 1 and the bond is priced at 98.630, it is profitable for the issuer to call the bond. The price of the bond in this case would be the call price of 98.

When the one-period spot rate is 11% in Period 1, it is not profitable for the issuer to exercise the call. The price of the bond in this case remains at 97.297

Note:

In the 9.5% case, the issuer could buy the bond back at 98 financed by issuing a one-year bond at 9.5% interest. One period later the issuer would owe 98 × (1.095) = 107.31; this represents a savings of 108-107.31 = 0.69.

Note, the value of that savings in period one is .69/1.095 = 0.63, which is equal to the difference between the bond price and the call price: 98.630 - 98 = .63.

Subtracting the call value of 0.2864 from the noncallable bond value of 96.330, we obtain a callable bond value of 96.044.

96.044 is the same value we obtained using the constraint approach.

$$S_0 = 10\%,\ B_0^{NC} = 96.330$$

$$V_0^C = \frac{.5(0)+.5(.630)}{(1.10)} = .2864$$

$$B_0^C = B_0^{NC} - V_0^C$$

$$= 96.330 - .2864 = 96.044$$

$$S_u = 11\%,\ B_u^{NC} = 97.297$$

$$V_u^C = Max[B_u - CP, 0]$$

$$V_u^C = Max[97.297 - 98, 0] = 0$$

$$B_u^C = B_u^{NC} - V_u^C$$

$$= 97.297 - 0 = 97.297$$

$$S_d = 9.5\%,\ B_d^{NC} = 98.630$$

$$V_d^C = Max[B_d - CP, 0]$$

$$V_d^C = Max[98.630 - 98, 0] = .630$$

$$B_d^C = B_d^{NC} - V_d^C$$

$$= 98.630 - .630 = 98$$

Valuing a 3-Period Option-Free Bond

The binomial approach to valuing a two-period bond requires only a one-period binomial tree of one-period spot rates.

If we want to value a three-period bond, we in turn need a two-period interest rate tree.

Example

Suppose we wanted to value a three-period, 9% coupon bond with no default risk or option features.

In this case, market risk exist in two periods: Period 3, where there are three possible spot rates, and Period 2, where there are two possible rates.

Valuing a 3-Period Option-Free Bond

To value the bond

We first determine the three possible values of the bond in Period 2 given the three possible spot rates and the bond's certain cash flow next period (maturity). The three possible values in Period 2 are:

$$B_{uu} = 109/1.121 = 97.2346,$$
$$B_{ud} = 109/1.1045 = 98.6872,$$
$$B_{dd} = 109/1.09025 = 99.9771$$

$$C = 9,\ F = \$100$$
$$S_0 = .10,\ u = 1.1,\ d = .95$$

$$S_0 = 10\%$$
$$B_0 = \frac{.5(96.3612 + 9) + .5(98.9335 + 9)}{(1.10)} = 96.9521$$

$$S_u = uS_0 = 11\%$$
$$B_u = \frac{.5(97.2346 + 9) + .5(98.6872 + 9)}{(1.11)} = 96.3612$$

$$S_d = dS_0 = 9.5\%$$
$$B_d = \frac{.5(96.6872 + 9) + .5(99.9771 + 9)}{(1.095)} = 98.9335$$

$$S_{uu} = u^2 S_0 = 12.1\%$$
$$B_{uu} = \frac{109}{1.121} = 97.2346$$

$$S_{ud} = udS_0 = 10.45\%$$
$$B_{ud} = \frac{109}{1.1045} = 98.6872$$

$$S_{dd} = d^2 S_0 = 9.025\%$$
$$B_{dd} = \frac{109}{1.09025} = 99.9771$$

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Valuing a 3-Period Option-Free Bond

Given these values, we next roll the tree to the first period and determine the two possible values there as the present values of the expected cash flows in Period 2:

$$B_u = \frac{.5[97.2346+9] + .5[98.6872+9]}{1.11} = 96.3612$$

$$B_d = \frac{.5[98.6872+9] + .5[99.9771+9]}{1.095} = 98.9335$$

Finally, using the bond values in Period 1, we roll the tree to the current period where we determine the value of the bond to be 96.9521:

$$B_0 = \frac{.5[96.3612+9] + .5[98.9335+9]]}{1.10} = 96.9521$$

Valuing a 3-Period Callable Bond

To determine the value of the bond given its callable:

1. First compare each of the noncallable bond values with the call price in Period 2 (one period from maturity); then take the minimum of the two as the callable bond value.

2. Next roll the callable bond values from Period 2 to Period 1 and determine the two bond values at each node as the present value of the expected cash flows, and then for each case select the minimum of the value calculated or the call price.

3. Finally, roll those two callable bond values to the current period and determine the callable bond's price as the present value of Period 1's expected cash flows.

Valuing a 3-Period Callable Bond

The exhibit on the next slide shows the binomial tree value of the three-period, 9% bond given a call feature with a CP = 98.

At the two lower nodes in Period 2, the bond would be called at 98 and therefore the callable bond price would be 98; at the top node in Period 2, the bond price of 97.2346 would prevail.

Rolling Period 2's prices to Period 1, the present values of the expected cash flows are 96.0516 at the 11% spot rate and 97.7169 at the 9.5% rate. Since neither of these values are less than the CP of 98, each represents the callable bond value at that node.

Rolling Period 1's two values to the current period, we obtain a value of 96.2584 for the three-period callable bond.

$$C = 9,\ F = \$100,\ CP = 98$$
$$S_0 = .10,\ u = 1.1,\ d = .95$$

$$S_0 = 10\%$$
$$B_0^C = \frac{.5(96.0516 + 9) + .5(97.7169 + 9)}{(1.10)} = 96.258$$

$$S_u = uS_0 = 11\%$$
$$B_u = \frac{.5(97.2346 + 9) + .5(98. + 9)}{(1.11)} = 96.0516$$
$$B_u^C = \mathrm{Min}[B_u, CP]$$
$$B_u^C = \mathrm{Min}[96.0516,\ 98] = 96.0516$$

$$S_d = dS_0 = 9.5\%$$
$$B_d = \frac{.5(98 + 9) + .5(98 + 9)}{(1.095)} = 97.7169$$
$$B_d^C = \mathrm{Min}[B_d,\ CP]$$
$$B_d^C = \mathrm{Min}[97.7169,\ 98] = 97.7169$$

$$S_{uu} = u^2 S_0 = 12.1\%$$
$$B_{uu} = \frac{109}{1.121} = 97.2346$$
$$B_{uu}^C = \mathrm{Min}[B_{uu}, CP]$$
$$B_{uu}^C = \mathrm{Min}[97.2346, 98] = 97.2346$$

$$S_{ud} = udS_0 = 10.45\%$$
$$B_{ud} = \frac{109}{1.1045} = 98.6872$$
$$B_{ud}^C = \mathrm{Min}[B_{ud},\ CP]$$
$$B_{ud}^C = \mathrm{Min}[98.6872, 98] = 98$$

$$S_{dd} = d^2 S_0 = 9.025\%$$
$$B_{dd} = \frac{109}{1.09025} = 99.9771$$
$$B_{dd}^C = \mathrm{Min}[B_{dd}, CP]$$
$$B_{dd}^C = \mathrm{Min}[99.9771,\ 98] = 98$$

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Valuing a 3-Period Callable Bond

The alternative approach to valuing the callable bond is to determine the value of the call option at each node and then subtract that value from the noncallable value to obtain the callable bond's price.

Note:

Different from our previous two-period case, when there are three periods or more, we need to take into account that prior to maturity the bond issuer has two choices:

1) Can either exercise the option, or
2) Can hold it for another period

Valuing a 3-Period Callable Bond

The exercising value, IV (intrinsic value), is:

$$\boxed{IV = Max[B_t^{NC} - CP, 0]}$$

The value of holding, V_H, is the present value of the expected call value next period:

$$\boxed{V_H = \frac{qV_u^C + (1-q)V_d^C}{1+S}}$$

Valuing a 3-Period Callable Bond

In Period 2 the value of holding is zero at all three nodes since next period is maturity where it is too late to call. The issuer, though, would find it profitable to exercise in two of the three cases where the call price is lower than the bond values. The three possible callable bond values in Period 2 are:

$$B_{uu}^{C} = B_{uu}^{NC} - V_{uu}^{C} = 97.2346 - Max[97.2346 - 98, 0] = 97.2346$$

$$B_{ud}^{C} = B_{ud}^{NC} - V_{ud}^{C} = 98.6872 - Max[98.6872 - 98, 0] = 98$$

$$B_{dd}^{C} = B_{dd}^{NC} - V_{dd}^{C} = 99.9771 - Max[99.9771 - 98, 0] = 98$$

Valuing a 3-Period Callable Bond

In Period 1 at the lower node:
The non-callable bond price is greater than the call price.
In this case, the IV is 98.9335 - 98 = .9335.
The value of holding the call, though, is 1.2166:

$$V_H = \frac{.5[Max[98.6872 - 98, 0]] + .5[Max[99.9771 - 98, 0]]}{1.095} = 1.2166$$

Thus, the issuer would find it more valuable to defer the exercise one period. As a result, the value of the call option is 1.2166 and the value of the callable bond is 97.7169

$$Max[IV, V_H] = Max[.8394, 1.2166] = 1.2166$$

$$B_d^C = B_d^{NC} - V_d^C = 98.9335 - 1.2166 = 97.7169$$

Valuing a 3-Period Callable Bond

In Period 1 at the upper node:
The price of the non-callable is 96.3612
The exercise value is 0.
The value of the call option in this case is equal to its holding value of .3095:

$$\boxed{V_H = \frac{.5[Max[97.2346-98,0]]+.5[Max[98.6872-98,0]]}{1.11} = .3095}$$

The value of the callable bond is 96.0517:

$$\boxed{B_u^C = B_u^{NC} - V_u^C = 96.3612 - .3095 = 96.0517}$$

Valuing a 3-Period Callable Bond

Current Period:

Rolling the two possible option values of .3095 and 1.2166 in Period 1 to the current period, we obtain the current value of the option of .6937 and the same callable bond value of 96.2584 that we obtained using the first approach:

$$V_0^C = \frac{.5[.3095] + .5[1.2166]}{1.10} = .6937$$

$$B_0^C = B_0^{NC} - V_0^C = 96.9521 - .6937 = 96.2584$$

$C = 9,\ F = \$100,\ CP = 98$

$S_0 = .10,\ u = 1.1,\ d = .95$

$S_0 = 10\%,\ B_0^{NC} = 96.9521$

$$V_0^C = V_H = \frac{.5(.3095) + .5(1.2166)}{(1.10)} = .6937$$

$$B_0^C = B_0^{NC} - V_0^C = 96.9521 - .6937 = 96.2584$$

$S_u = 11\%,\ B_u^{NC} = 96.3612$

$$V_H = \frac{.5(0) + .5(.6872)}{(1.11)} = .3095$$

$$IV = Max[B_u^{NC} - CP,\ 0] = Max[96.3612 - 98, 0] = 0$$

$$V_u^C = Max[V_H, IV] = Max[.3095, 0] = .3095$$

$$B_u^C = B_u^{NC} - V_u^C = 96.3612 - .3095 = 96.0516$$

$S_d = 9.5\%,\ B_d^{NC} = 98.9335$

$$V_H = \frac{.5(.6872) + .5(1.9771)}{(1.095)} = 1.2166$$

$$IV = Max[B_d^{NC} - CP,\ 0] = Max[98.9335 - 98, 0] = .9335$$

$$V_d^C = Max[V_H, IV] = Max[1.2166,\ .9335] = 1.2166$$

$$B_d^C = B_d^{NC} - V_d^C = 98.9335 - 1.2166 = 97.7169$$

$S_{uu} = 12.1\%,\ B_{uu}^{NC} = 97.2346$

$$IV = Max[B_{uu}^{NC} - CP, 0] = Max[97.2346 - 98, 0] = 0$$

$$V_{uu}^C = IV = 0$$

$$B_{uu}^C = B_{uu}^{NC} - V_{uu}^C = 97.2346 - 0 = 97.2346$$

$S_{ud} = 10.45\%,\ B_{ud}^{NC} = 98.6872$

$$IV = Max[B_{ud}^{NC} - CP, 0] = Max[98.6872 - 98, 0] = .6872$$

$$V_{ud}^C = IV = .6872$$

$$B_{ud}^C = B_{ud}^{NC} - V_{ud}^C = 98.6872 - .6872 = 98$$

$S_{dd} = 9.025\%,\ B_{dd}^{NC} = 99.9771$

$$IV = Max[B_{dd}^{NC} - CP, 0] = Max[99.9771 - 98, 0] = 1.9771$$

$$V_{dd}^C = IV = 1.9771$$

$$B_{dd}^C = B_{dd}^{NC} - V_{dd}^C = 99.9771 - 1.9771 = 98$$

Other Embedded Option Features

The binomial tree can be extended to the valuation of bonds with other embedded option features

Other embedded options include:

- Put Option

- Sinking fund arrangement in which the issuer has the option to buy some of the bonds back either at their market price or at a call price

- Sinking fund Stock convertibility clause

Putable Bond

A putable bond, or put bond, gives the holder the right to sell the bond back to the issuer at a specified exercise price (or put price), PP

In contrast to callable bonds, putable bonds benefit the holder:

If the price of the bond decreases below the exercise price, then the bondholder can sell the bond back to the issuer at the exercise price.

Putable Bond

From the bondholder’s perspective, a put option provides a hedge against a decrease in the bond price.

If rates decrease in the market, then the bondholders benefits from the resulting higher bond prices.
If rates increase, then the bondholder can exercise, giving the downside protection.

Given that the bondholder has the right to exercise, the price of a putable bond will be equal to the price of an otherwise identical non-putable bond plus the value of the put option (V_0^P):

$$\boxed{B_0^P = B_0^{NP} + V_0^P}$$

Binomial Valuation of Putable Bond

Since the bondholder will find it profitable to exercise whenever the put price exceeds the bond price, the value of a putable bond can be found using the binomial approach by comparing bond prices at each node with the put price and selecting the maximum of the two:

$$\boxed{B_t^P = \mathrm{Max}[B_t, PP]}$$

Binomial Valuation of Putable Bond

The same binomial value can also be found by determining the value of the put option at each node and then pricing the putable bond as the value of an otherwise identical non-putable bond plus the value of the put option.

In using the second approach, the value of the put option will be the maximum of either its intrinsic value (or exercising value):

$$IV = Max[PP - B_t, 0]$$

or its holding value (the present value of the expected put value next period).

In most cases, though, the put's intrinsic value will be greater than its holding value.

Valuing a 3-Period Putable Bond

Suppose the three-period, 9% option-free bond in our previous example had a put option giving the bondholder the right to sell the bond back to the issuer at an exercise price of PP = 97 in Periods 1 or 2.

Period 2:

Using the two-period tree of one-period spot rates and the corresponding bond values for the option-free bond we start at Period 2 and investigate each of the nodes to determine if there is an advantage for the holder to exercise.

In all three of the cases in Period 2, the bond price exceeds the exercise price; thus, there are no possible exercise advantages in this period and each of the possible prices of the putable bond are equal to their nonputable values and the values of each of the put options are zero.

Valuing a 3-Period Putable Bond

Period 1 Upper Node:

In Period 1 it is profitable for the holder to exercise when the spot rate is 11%. At that node, the value of the non-putable bond is 96.3612, compared to PP = 97; thus the value of putable bond is its exercise price of 97:

$$\boxed{B_u^P = Max[96.3612, 97] = 97}$$

The putable bond price of 97 can also be found by subtracting the value of the put option from the price of the non-putable bond. The value of the put option at this node is 0.6388; thus, the value of the putable bond is 97:

$$\boxed{V_u^P = Max[IV, V_H] = Max[97 - 96.3612,\ 0] = .6388}$$

$$\boxed{\begin{aligned} B_u^P &= B_u^{NP} + V_u^P \\ B_u^P &= 96.3612 + .6388 = 97 \end{aligned}}$$

Valuing a 3-Period Putable Bond

Period 1 Lower Node:

At the lower node in Period 1, it is not profitable to exercise nor is there any holding value of the put option since there is no exercise advantage in Period 2. Thus at the lower node, the non-putable bond price prevails.

Current Period

- Rolling the two putable bonds values in Period 1 to the present, we obtain a current value of the putable bond of 97.2425:

$$\boxed{B_0^P = \frac{.5[97+9] \;\; + .5[98.9335+9]}{1.10} = 97.2425}$$

Valuing a 3-Period Putable Bond

Current Period

The current value also can be obtained using the alternative approach by computing the present value of the expected put option value in Period 1 and then adding that to the current value of the non-putable bond.

With possible exercise values of 0.6388 and 0 in Period 1, the current put option value is 0.2904:

$$\boxed{V_0^P = \frac{.5[.6388] + .5[0]}{1.10} = .2904}$$

$$\boxed{B_0^P = B_0^{NP} + V_0^P = 96.9521 + .2904 = 97.2425}$$

$$C = 9,\ F = \$100,\ PP = 97$$
$$S_0 = .10,\ u = 1.1,\ d = .95$$

$$S_0 = 10\%$$
$$B_0^P = \frac{.5(97+9)+.5(98.9335+9)}{(1.10)} = 97.2425$$

$$S_u = uS_0 = 11\%$$
$$B_u = \frac{.5(97.2346+9)+.5(98.6872+9)}{(1.11)} = 96.3612$$
$$B_u^P = \text{Max}[B_u, PP]$$
$$B_u^P = \text{Max}[96.3612,\ 97] = 97$$

$$S_d = dS_0 = 9.5\%$$
$$B_d = \frac{.5(98.6872+9)+.5(99.9771+9)}{(1.095)} = 98.9335$$
$$B_d^P = \text{Max}[B_d,\ PP]$$
$$B_d^P = \text{Min}[98.9334,\ 97] = 98.9335$$

$$S_{uu} = u^2S_0 = 12.1\%$$
$$B_{uu} = \frac{109}{1.121} = 97.2346$$
$$B_{uu}^P = \text{Max}[B_{uu}, PP]$$
$$B_{uu}^P = \text{Max}[97.2346, 97] = 97.2346$$

$$S_{ud} = udS_0 = 10.45\%$$
$$B_{ud} = \frac{109}{1.1045} = 98.6872$$
$$B_{ud}^P = \text{Max}[B_{ud},\ PP]$$
$$B_{ud}^P = \text{Max}[98.6872, 97] = 98.6872$$

$$S_{dd} = d^2S_0 = 9.025\%$$
$$B_{dd} = \frac{109}{1.09025} = 99.9771$$
$$B_{dd}^P = \text{Max}[B_{dd}, PP]$$
$$B_{dd}^P = \text{Max}[99.9771,\ 97] = 99.9771$$

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Sinking-Fund Bonds

Many bonds have sinking fund clauses specified in their indenture requiring that the issuer make scheduled payments into a fund or buy up a certain proportion of the bond issue each period.

Often when the sinking fund agreement specifies an orderly retirement of the issue, the issuer is given an option of either purchasing the bonds in the market or calling the bonds at a specified call price.

The call option embedded in the sinking fund agreement makes the sinking fund valuable to the issuer.

- If interest rates are relatively high, then the issuer will be able to buy back the requisite amount of bonds at a relatively low market price
- If rates are low and the bond price high, though, then the issuer will be able to buy back the bonds on the call option at the call price

Thus, a sinking fund bond with this type of call provision should trade at a lower price than an otherwise identical non-sinking fund bond

Binomial Valuation of a Sinking-Fund Bond

Example:

Suppose a company issues a $15M, three-period bond with a sinking fund obligation requiring:

- The issuer sink $5M of face value after the first period and $5M after the second.
- The issuer has an option of either buying the bonds in the market or calling them at a call price of 98.

Assume the same interest rate tree and bond values characterizing the three-period, 9% non-callable bond described previously apply to this bond without its sinking fund agreement.

Binomial Valuation of a Sinking-Fund Bond

With the sinking fund, the issuer has two options:

1. At the end of Period 1, the issuer can buy \$5M worth of the bond either at 98 or at the bond's market price.
2. At the end of Period 2, the issuer has another option to buy \$5M worth of the bond either at 98 or the market price.

As shown in the next two exhibits:

1. The value of the Period 1call option (in terms of \$100 face value) is $V_0^{SF(1)} = 0.4243$
2. The value of the Period 2 call option is $V_0^{SF(2)} = 0.6937$.

Binomial Valuation of a Sinking-Fund Bond

Note:

Since the sinking fund arrangement requires an immediate exercise or bond purchase at the specified sinking fund dates, the possible values of the sinking fund's call features at those dates are equal to their intrinsic values.

This differs from the valuation of a standard callable bond where a holding value is also considered in determining the value of the call option.

Value of Sinking Fund
Call in Period 1
Sinking Fund CP = 98

$S_u = 11\%$

$B_u^{NSF} = 96.3612$

$IV = Max[B_u^{NSF} - CP, 0]$

$= Max[96.3612 - 98, 0] = 0$

$S_0 = 10\%,$

$$V_0^{SF(1)} = \frac{.5(0) + .5(.9335)}{(1.10)} = .4243$$

$S_d = 9.5\%$

$B_d^{NSF} = 98.9335$

$IV = Max[B_d^{NSF} - CP, 0]$

$= Max[98.9335 - 98, 0] = .9335$

Value of Sinking Fund
Call in Period 2
Sinking Fund CP = 98

$$S_0 = 10\%,\ B_0^{NSF} = 96.9521$$
$$V_0^{SF(2)} = \frac{.5(.3095)+.5(1.2166)}{(1.10)} = .6937$$

$$S_u = 11\%,\ B_u^{NSF} = 96.3612$$
$$V_u^{SF(2)} = \frac{.5(0)+.5(.6872)}{(1.11)} = .3095$$

$$S_d = 9.5\%,\ B_d^{NSF} = 98.9335$$
$$V_d^{SF(2)} = \frac{.5(.6872)+.5(1.9771)}{(1.095)} = 1.2166$$

$$S_{uu} = 12.1\%,\ B_{uu}^{NSF} = 97.2346$$
$$IV = Max[B_{uu}^{NSF} - CP, 0] = Max[97.2346 - 98, 0] = 0$$

$$S_{ud} = 10.45\%,\ B_{ud}^{NSF} = 98.6872$$
$$IV = Max[B_{ud}^{NSF} - CP, 0] = Max[98.6872 - 98, 0] = .6872$$

$$S_{dd} = 9.025\%,\ B_{dd}^{NSF} = 99.9771$$
$$IV = Max[B_{dd}^{NSF} - CP, 0] = Max[99.9771 - 98, 0] = 1.9771$$

Binomial Valuation of a Sinking-Fund Bond

Since each option represents 1/3 of the issue, the value of the bond's sinking fund option is 0.3727, and the value of the sinking fund bond is 96.5794 per $100 face value. Thus, the total value of the $15M face value issue is $14.48691M:

$$\boxed{V_0^{SF} = (1/3)(.4243) + (1/3)(.6937) = .3727}$$

$$B_0^{SF} = B_0^{NSF} - V_0^{SF}$$

$$B_0^{SF} = 96.9521 - .3727 = 96.5794$$

$$\boxed{\text{Issue Value} = \frac{96.5794}{100}\$15\text{M} = \$14.48691\text{M}}$$

Like a standard callable bond, a sinking fund provision with a call feature lowers the value of an otherwise identical non-sinking fund bond.

Convertible Bonds

A convertible bond gives the holder the right to convert the bond into a specified number of shares of stock.

Convertibles are often sold as a subordinate issue, with the conversion feature serving as a bond sweetener.

To the investor, convertible bonds offer the potential for a high rate of return if the company does well and its stock price increases, while providing some downside protection as a bond if the stock declines.

Convertibles are usually callable, with the convertible bondholder usually having the right to convert the bond to stock if the issuer does call.

Convertible Bonds: Terms

Consider a three-period, 9% bond convertible into four shares of the underlying company's stock.

The conversion ratio (CR) is the number of shares of stock that can be converted when the bond is tendered for conversion. The conversion ratio for this bond is four.

The conversion value (CV) is the convertible bond's value as a stock. At a given point in time, the conversion value is equal to the conversion ratio times the market price of the stock (P_t^S):

If the current price of the stock were 92, then the bond's conversion value would be CV = (4) × ($92) = $368.

$$\boxed{CV_t = (CR)\, P_t^S}$$

Convertible Bonds: Terms

The straight debt value (SDV) is the convertible bond's value as a nonconvertible bond. This value is obtained by discounting the convertible's cash flows by the discount rate on a comparable non-convertible bond.

The conversion price is the bond's par value divided by the conversion ratio: F/CR.

Convertible Bonds: Minimum Price

Arbitrage ensures that the minimum price of a convertible bond is the greater of either its straight debt value or its conversion value:

$$\boxed{\mathrm{Min}\,B_t^{CB} = \mathrm{Max}[CV_t, SDV_t]}$$

If a convertible bond is priced below its conversion value, arbitrageurs could buy it, convert it to stock, and then sell the stock in the market to earn a riskless profit. Arbitrageurs seeking such opportunities would push the price of the convertible up until it is at least equal to its CV.

Similarly, if a convertible is selling below its straight debt value (SDV), then arbitrageurs could profit by buying the convertible and selling it as a regular bond.

Convertible Bonds: Maximum Price

If the convertible is callable, the call price at which the issuer can redeem the bond places a maximum limit on the convertible.

That is, the issuer will find it profitable to buy back the convertible bond once its price is equal to the call price.

Buying back the bond, in turn, frees the company to sell new stock or bonds at prices higher than the stock or straight debt values associated with the convertible.

Thus, the maximum price of a convertible is the call price.

The actual price that a convertible will trade for will be at a premium above its minimum value but below its maximum.

Binomial Valuation of Convertible Bonds

The valuation of a convertible bond with an embedded call option is more difficult than the valuation of a bond with just one option feature.

The valuation of convertibles needs to take into account:

1. The random patterns of interest rates
2. The random patterns of the stock's price
3. The correlation between interest rates and the stock price

Binomial Valuation of Convertible Bonds

Example:
Three-period, 10% convertible bond
Face value = 1000
Convertible into to 10 shares of the underlying company's stock
(CR = 10).

Assume:
The bond has no call option and no default risk.
The current yield curve is flat at 5%.
The yield curve will stay at 5% for the duration of the three periods (i.e. no market risk)

Binomial Valuation of Convertible Bonds

Assume:

The convertible bond's underlying stock price follows a binomial process where in each period it has an equal chance it can either increase to equal u times its initial value or decrease to equal d times the initial value, where,

$u = 1.1$

$d = 1/1.1 = 0.9091$

q = probability of stock increasing one period = 0.5

Current stock price is 92

The possible stock prices resulting from this binomial process and the convertible bond's conversion values are shown in the next exhibit.

At the maturity date (end of Period 3), the bondholder will have a coupon worth 100 and will either convert the bond to stock or receive the principal of 1000.

Binomial Valuation of Convertible Bonds

In Period 3 at 122.45:

The convertible bondholder would exercise her option, converting the bond to ten shares of stock. The value of the convertible bond, B^{CB}, would therefore be equal to its conversion value of 1224.50 plus the $100 coupon:

$$\boxed{\begin{aligned} B^{CB}_{uuu} &= \mathrm{Max}[CV_t, F] + C \\ B^{CB}_{uuu} &= \mathrm{Max}[1224.50, 1000] + 100 \\ B^{CB}_{uuu} &= 1324.50 \end{aligned}}$$

In Period 3 at 101.20:

The bondholder would find it profitable to convert; thus, the value of the convertible in this case would be its conversion value of 1012 plus the $100 coupon.

$$\boxed{\begin{aligned} B^{CB}_{uud} &= \mathrm{Max}[CV_t, F] + C \\ B^{CB}_{uud} &= \mathrm{Max}[1012, 1000] + 100 \\ B^{CB}_{uud} &= 1120 \end{aligned}}$$

Binomial Valuation of Convertible Bonds

In Period 3 at 83.64 and 69.12:

The conversion is worthless; thus the value of the convertible bond is equal to the principal plus the coupon: 1100.

$$\boxed{B_{udd}^{CB} = Max[836.40, 1000] + 100 = 1100}$$

$$\boxed{B_{ddd}^{CB} = Max[691.20, 1000] + 100 = 1100}$$

Binomial Valuation of Convertible Bonds

In Period 2 at each node:

The value of the convertible bond is equal to the maximum of either the present value of the bond's expected value at maturity or its conversion value.

At all three stock prices, the present values of the bond's expected values next period are greater than the bond's conversion values.

Example:

At P^S_{uu} = 111.32, the CV is 1113.20 compared to the convertible bond value of 1160.24; thus the value of the convertible bond is 1160.24:

$$B_{uu} = \frac{.5[1324.50] + .5[1112]}{1.05} = 1160.24$$

$$B^{CB}_{uu} = \mathrm{Max}[B_{uu}, CV] = [1160.24, 1113.20] = 1160.24$$

Binomial Valuation of Convertible Bonds

In Period 1 at each node:

The two possible bond values in Period 1 (generated by rolling the three convertible bond values in Period 2 to Period 1) also exceed their conversion values.

The Current Period

Rolling Period 1 values to the current period, we obtain a convertible bond value of 1164.29.

Stock Price $= P_0^S = 92$, $u = 1.10$
$d = .9091$, Spot Rate $= 5\%$,
Convertible : Coupon $= 100$, $F = 1000$,
Conversion Ratio $= 10$

$P_0^S = 92,\ S_0 = 5\%$
$CV = 920$
$$B_0^{CB} = \frac{.5(1149.32 + 100) + .5(1095.69 + 100)}{(1.05)} = 1164.29$$

$P_u^S = 101.20,\ S_u = 5\%$
$CV = 1012$
$$B_u = \frac{.5(1160.24 + 100) + .5(1053.33 + 100)}{(1.05)} = 1149.64$$
$B_u^{CB} = Max[CV, B_u] = 1149.32$

$P_d^S = 83.64,\ S_d = 5\%$
$CV = 836.40$
$$B_d = \frac{.5(1053.33 + 100) + .5(1047.62 + 100)}{(1.05)} = 1095.69$$
$B_d^{CB} = Max[CV, B_d] = 1095.69$

$P_{uu}^S = 111.32,\ S_{uu} = 5\%$
$CV = 1113.20$
$$B_{uu} = \frac{.5(1324.50) + .5(1112)}{(1.05)} = 1160.24$$
$B_{uu}^{CB} = Max[CV, B_{uu}] = 1160.24$

$P_{ud}^S = 92,\ S_{ud} = 5\%$
$CV = 920$
$$B_{ud} = \frac{.5(1112) + .5(1100)}{(1.05)} = 1053.33$$
$B_{ud}^{CB} = Max[CV, B_{ud}] = 1053.33$

$P_{dd}^S = 76.03,\ S_{dd} = 5\%$
$CV = 760.30$
$$B_{dd} = \frac{1100}{(1.05)} = 1047.62$$
$B_{dd}^{CB} = Max[CV, B_{dd}] = 1047.62$

$P_{uuu}^S = 122.45$
$CV = 1224.50$
$F = 1000$
$C = 100$
$B_{uuu}^{CB} = Max[CV, F] + C = 1324.50$

$P_{uud}^S = 101.20$
$CV = 1012$
$F = 1000$
$C = 100$
$B_{uud}^{CB} = Max[CV, F] + C = 1112$

$P_{udd}^S = 83.64$
$CV = 836.40$
$F = 1000$
$C = 100$
$B_{udd}^{CB} = Max[CV, F] + C = 1100$

$P_{ddd}^S = 69.12$
$CV = 691.20$
$F = 1000$
$C = 100$
$B_{ddd}^{CB} = Max[CV, F] + C = 1100$

Binomial Valuation of Callable Convertible Bonds

With a callable convertible bond, the issuer will find it profitable to call the convertible prior to maturity whenever the price of the convertible is greater than the call price.

When the convertible bondholder is faced with a call, usually has the choice of either tendering the bond at the call price or converting it to stock.

Since the issuer will call whenever the call price exceeds the convertible bond price, he is in effect forcing the holder to convert.

By doing this, the issuer takes away the bondholder's value of holding the convertible, forcing the convertible bond price to equal its conversion value.

Binomial Valuation of Callable Convertible Bonds

Suppose the convertible bond is callable in Periods 1 and 2 at CP = 1100.

In Period 2 at 111.32

- The conversion value is 1113.20.
- In this case, the issuer can force the bondholder to convert by calling the bond.
- The call option therefore reduces the value of the convertible from 1160.24 to 1113.20.

In Period 2 at 92 and 76.03

- Neither conversion by the bondholders or calling by the issuer is economical.
- Thus the bond values prevail.

Binomial Valuation of Callable Convertible Bonds

In Period 1 at 102.20

- The call price of 1100 is below the bond value (1126.92), but above the conversion value (1012).
- In this case, the issuer would call the bond and the holder would take the call instead of converting.
- The value of the callable convertible bond in this case would be the call price of 1100.

In Period 1 at 83.64

- Calling and converting are not economical. The bond value of 1095.69 prevails.

Current Period

- Rolling Period 1's upper and lower convertible bond values to the current period, we obtain a value for the callable convertible bond of 1140.80.
- The callable convertible bond value is less than the non-callable convertible bond value of 1164.29.

$$\text{Stock Price} = P_0^S = 92,\ u = 1.10$$
$$d = .9091,\ \text{Spot Rate} = 5\%,$$
$$\text{Convertible: Coupon} = 100,\ F = 1000,$$
$$\text{Conversion Ratio} = 10,\ \text{Callable at } CP = 1100$$

$$P_0^S = 92,\ S_0 = 5\%$$
$$CV = 920$$
$$B_0^{CB} = \frac{.5(1100 + 100) + .5(1095.69 + 100)}{(1.05)} = 1140.80$$

$$P_u^S = 101.20,\ S_u = 5\%$$
$$CV = 1012,\ CP = 1100$$
$$B_u = \frac{.5(1113.2 + 100) + .5(1053.33 + 100)}{(1.05)} = 1126.92$$
$$B_u^{CB} = \mathrm{Max}[CV, \mathrm{Min}[B_u, CP]] = 1100$$

$$P_d^S = 83.64,\ S_d = 5\%$$
$$CV = 836.40,\ CP = 1100$$
$$B_d = \frac{.5(1053.33 + 100) + .5(1047.62 + 100)}{(1.05)} = 1095.69$$
$$B_d^{CB} = \mathrm{Max}[CV,\ \mathrm{Min}[B_d, CP]] = 1095.69$$

$$P_{uu}^S = 111.32,\ S_{uu} = 5\%$$
$$CV = 1113.20,\ CP = 1100$$
$$B_{uu} = \frac{.5(1324.50) + .5(1112)}{(1.05)} = 1160.24$$
$$B_{uu}^{CB} = \mathrm{Max}[CV, \mathrm{Min}[B_{uu}, CP]] = 1113.20$$

$$P_{ud}^S = 92,\ S_{ud} = 5\%$$
$$CV = 920,\ CP = 1100$$
$$B_{ud} = \frac{.5(1112) + .5(1100)}{(1.05)} = 1053.33$$
$$B_{ud}^{CB} = \mathrm{Max}[CV, \mathrm{Min}[B_{ud}, CP]] = 1053.33$$

$$P_{dd}^S = 76.03,\ S_{dd} = 5\%$$
$$CV = 760.30,\ CP = 1100$$
$$B_{dd} = \frac{1100}{(1.05)} = 1047.62$$
$$B_{dd}^{CB} = \mathrm{Max}[CV, \mathrm{Min}[B_{dd}, CP] = 1047.62$$

$$P_{uuu}^S = 122.45$$
$$CV = 1224.50$$
$$F = 1000$$
$$C = 100$$
$$B_{uuu}^{CB} = \mathrm{Max}[CV, F] + C = 1324.50$$

$$P_{uud}^S = 101.20$$
$$CV = 1012$$
$$F = 1000$$
$$C = 100$$
$$B_{uud}^{CB} = \mathrm{Max}[CV, F] + C = 1112$$

$$P_{udd}^S = 83.64$$
$$CV = 836.40$$
$$F = 1000$$
$$C = 100$$
$$B_{udd}^{CB} = \mathrm{Max}[CV, F] + C = 1100$$

$$P_{ddd}^S = 69.12$$
$$CV = 691.20$$
$$F = 1000$$
$$C = 100$$
$$B_{ddd}^{CB} = \mathrm{Max}[CV, F] + C = 1100$$

Binomial Valuation of Callable Convertible Bonds

(with Different Interest Rates)

Suppose we assume uncertain spot rates.

To model uncertain stock prices and spot rates:

- Use correlation or regression analysis to first estimate the relationship between a stock's price and the spot rate.
- Given a binomial model of spot rates, identify the corresponding stock prices.
- Or given a binomial model of stock prices, identify the corresponding spot rates.

Binomial Valuation of Callable Convertible Bonds

(with Different Interest Rates)

Example:

Suppose using regression analysis, we estimated the following relationship between the stock in our above example and the one-period spot rate:

$$\boxed{S_t = .16 - .001P_t^S}$$

Using this equation, the corresponding spot rates associated with the stock prices from the three-period tree would be:

P_t^S	S_t
111.32	4.87%
101.20	5.90%
92.00	6.80%
83.64	7.64%
76.03	8.40%

Binomial Valuation of Callable Convertible Bonds

(with Different Interest Rates)

The next exhibit shows the binomial tree of stock prices along with their corresponding spot rates given the spot rate and stock price relation.

The convertible bond value = 1097.99.

This value is lower than the previous case in which we assumed a constant yield curve at 5%.

Stock Price $= P_0^S = 92$,
$u = 1.10$, $d = .9091$,
Convertible: Coupon $= 100$, $F = 1000$,
Conversion Ratio $= 10$, Callable at $CP = 1100$

$P_0^S = 92$, $S_0 = 6.8\%$
$CV = 920$
$B_0^{CB} = \dfrac{.5(1100 + 100) + .5(1045.31 + 100)}{(1.068)}$
$= 1097.99$

$P_u^S = 101.20$, $S_u = 5.9\%$
$CV = 1012$, $CP = 1100$
$B_u = \dfrac{.5(1113.20 + 100) + .5(1035.58 + 100)}{(1.059)}$
$= 1108.96$
$B_u^{CB} = Max[CV, Min[B_u, CP]] = 1100$

$P_d^S = 83.64$, $S_d = 7.64\%$
$CV = 836.40$, $CP = 1100$
$B_d = \dfrac{.5(1035.58 + 100) + .5(1014.76 + 100)}{(1.0764)}$
$= 1045.31$
$B_d^{CB} = Max[CV, Min[B_d, CP]] = 1045.31$

$P_{uu}^S = 111.32$, $S_{uu} = 4.87\%$
$CV = 1113.20$, $CP = 1100$
$B_{uu} = \dfrac{.5(1324.50) + .5(1112)}{(1.0487)}$
$= 1161.68$
$B_{uu}^{CB} = Max[CV, Min[B_{uu}, CP]] = 1113.20$

$P_{ud}^S = 92$, $S_{ud} = 6.8\%$
$CV = 920$, $CP = 1100$
$B_{ud} = \dfrac{.5(1112) + .5(1100)}{(1.068)}$
$= 1035.58$
$B_{ud}^{CB} = Max[CV, Min[B_{ud}, CP]] = 1035.58$

$P_{dd}^S = 76.03$, $S_{dd} = 8.4\%$
$CV = 760.30$, $CP = 1100$
$B_{dd} = \dfrac{1100}{(1.084)}$
$= 1014.76$
$B_{dd}^{CB} = Max[CV, Min[B_{dd}, CP] = 1014.76$

$P_{uuu}^S = 122.45$
$CV = 1224.50$
$F = 1000$
$C = 100$
$B_{uuu}^{CB} = Max[CV, F] + C$
$= 1324.50$

$P_{uud}^S = 101.20$
$CV = 1012$
$F = 1000$
$C = 100$
$B_{uud}^{CB} = Max[CV, F] + C$
$= 1112$

$P_{udd}^S = 83.64$
$CV = 836.40$
$F = 1000$
$C = 100$
$B_{udd}^{CB} = Max[CV, F] + C$
$= 1100$

$P_{ddd}^S = 69.12$
$CV = 691.20$
$F = 1000$
$C = 100$
$B_{ddd}^{CB} = Max[CV, F] + C$
$= 1100$

Binomial Valuation of Callable Convertible Bonds

(with Different Interest Rates)

It should be noted that modeling a bond with multiple option features and influenced by the random patterns of more than one factor is more complex in practice than the simple model described above.

The above model is intended only to provide some insight into the dynamics involved in valuing a bond with embedded convertible and call options given different interest rate and stock price scenarios.