# FINA 1082 Financial Management
# FINA 1098 Financial Management (Dual Award)


Dr Cesario MATEUS
Senior Lecturer in Finance and Banking
Room QA259 – Department of Accounting and Finance

c.mateus@greenwich.ac.uk
www.cesariomateus.com

# Session 4

Risk and Return

Portfolio Theory

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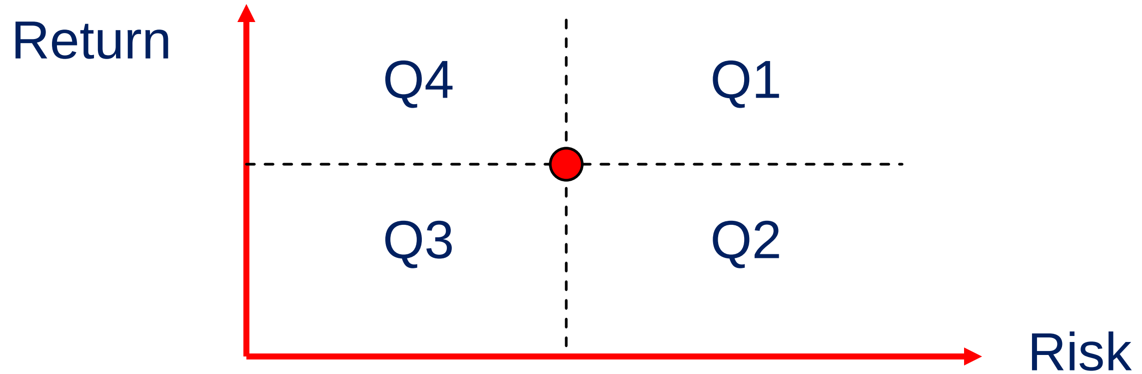
# Risk and Return

## Technicalities to start from

# Facts about Risk and Return

- Concepts defined from the perspective of investor not issuer.
- Assessment of risk and return represent the central issue for investment management
- Investors are risk averse (like returns and dislike risk)



# Risk and Risk Premium

Every investment involves some degree of uncertainty

- future selling price is unknown, future dividends are unknown, unknown future cash flows, …
- might have to sell assets due to emergency
- reinvestment rate might change (fall)
- increase in inflation changes the purchasing power of money / investment receipts

Expected holding period return (t = 0) is not the same as actual holding period return (t = 1)

$$E_t H_{t+1} = H_{t+1} + e_{t+1}$$

# Scenario Analysis and Probability Distribution

To quantify risk, address two questions:

- What Holding period return is possible ?
- How likely is it ?

Scenario Analysis

- Assess different economic scenarios (outcomes)

Probability Distribution

- Assign probabilities to possible outcomes

# Example : Scenario Analysis

| Measurement of Equity Returns | | | |
|---|---|---|---|
| State of the Economy | Probability (prob) | Expected Return ($E_R$) | Prob.×$E_R$ |
| Bad times | ¼ | -10% | -2.5% |
| Normal | ½ | 10% | 5% |
| Good times | ¼ | 20% | 5% |
| Expected Return = 7.5% | | | |

# Dispersion / Variability

Analyzing the data from the previous table, what surprises await us?

Definition of "surprise":

Surprise = Return - Expected Return

Measures of dispersion of actual return from expected return: variance (defined as average square surprises) or standard deviation (square root of variance).

The reason why we use expected (mean) return and standard deviation as return and risk measures in investment decision making process is because we assume that past asset returns follow normal distribution.

# Converting Prices into Rate of Returns

Financial data is usually reported as prices (Bloomberg, Datastream, etc)

For our statistical analysis and to be able to compare different investments we have to convert prices into returns

Example :

Suppose two shares with the following share price

- ABC share price (P = £1.10)
- XYZ share price (P = £4.35)
- Difficult to compare as they are measured in different 'units' (different base).

Converting Prices $\rightarrow$ Returns

- Arithmetic Rate of Return

  $R_t = (P_t + D_{t-1} - P_{t-1}) / P_{t-1}$

- Geometric Rate of Return (continuously compounded return).

  $R_t = \ln [(P_t + D_{t-1})/P_{t-1}]$

# Converting Prices into Returns - Example

Suppose that stock ABC had the following end-of-month prices:

245p on 31st August 2011

256p on 30th September 2011

Anticipated dividend for 2009 is 10p, therefore on a monthly basis it will be 10p/12=0.83333p

The arithmetic monthly rate of return of this stock is:

$R_t = (256 + 0.833 – 245)/245 = 0.048 = 4.8\%$

The geometric monthly rate of return of this stock is:

$R_t = \ln [(256 + 0.833)/ 245] =0.047 = 4.7\%$

# Advantages of Continuously Compounded Rate of Return

Geometric rate of return is also known as continuously compounded return.

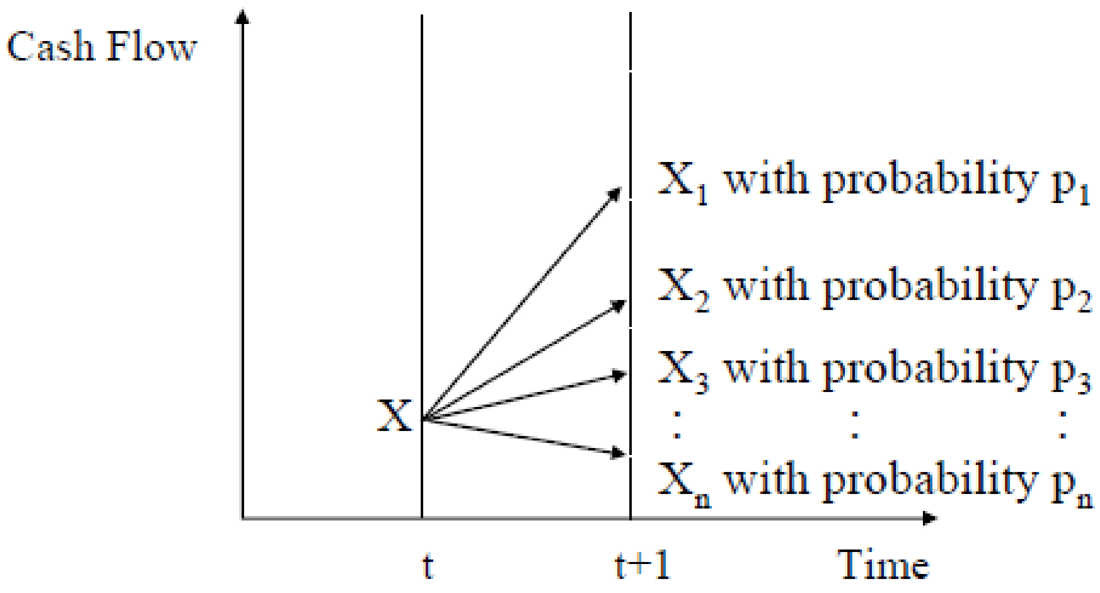
Differences of calculating the arithmetic or continuously compounded rate of return are small, especially for daily, weekly or even monthly data

If the formula for Geometric return is solved for P, continuously compounded rate of return will never give negative prices.

# The Probability Distribution Approach

We assume that investors can specify the possible outcomes ($X_1$ through $X_n$) and associate probabilities or likelihoods ($p_1$ through $p_n$) with these outcomes

Note: $X_1$ through $X_n$ are not known with certainty



# The Probability Distribution Approach

We assume that investors can specify the possible outcomes ($X_1$ through $X_n$) and associate probabilities or likelihoods ($p_1$ through $p_n$) with these outcomes. For each state, convert the cash flows into rates of returns

| State | Probability | Cash Flows | Rate of Return |
|---|---|---|---|
| 1 | $p_1$ | $X_1$ | $R_1 = (X_1 - X)/X$ |
| 2 | $p_2$ | $X_2$ | $R_2 = (X_2 - X)/X$ |
| 3 | $p_3$ | $X_3$ | $R_3 = (X_3 - X)/X$ |
| : | : | : | : |
| $n$ | $p_n$ | $X_n$ | $R_n = (X_n - X)/X$ |

Note: $p_1 + p_2 + ... + p_n = 1$, because one of these states of the world will be realized

## Interpreting Return and Risk Measures

- A general interpretation of the expected return and standard deviation of return requires assuming returns are continuously and normally distributed.

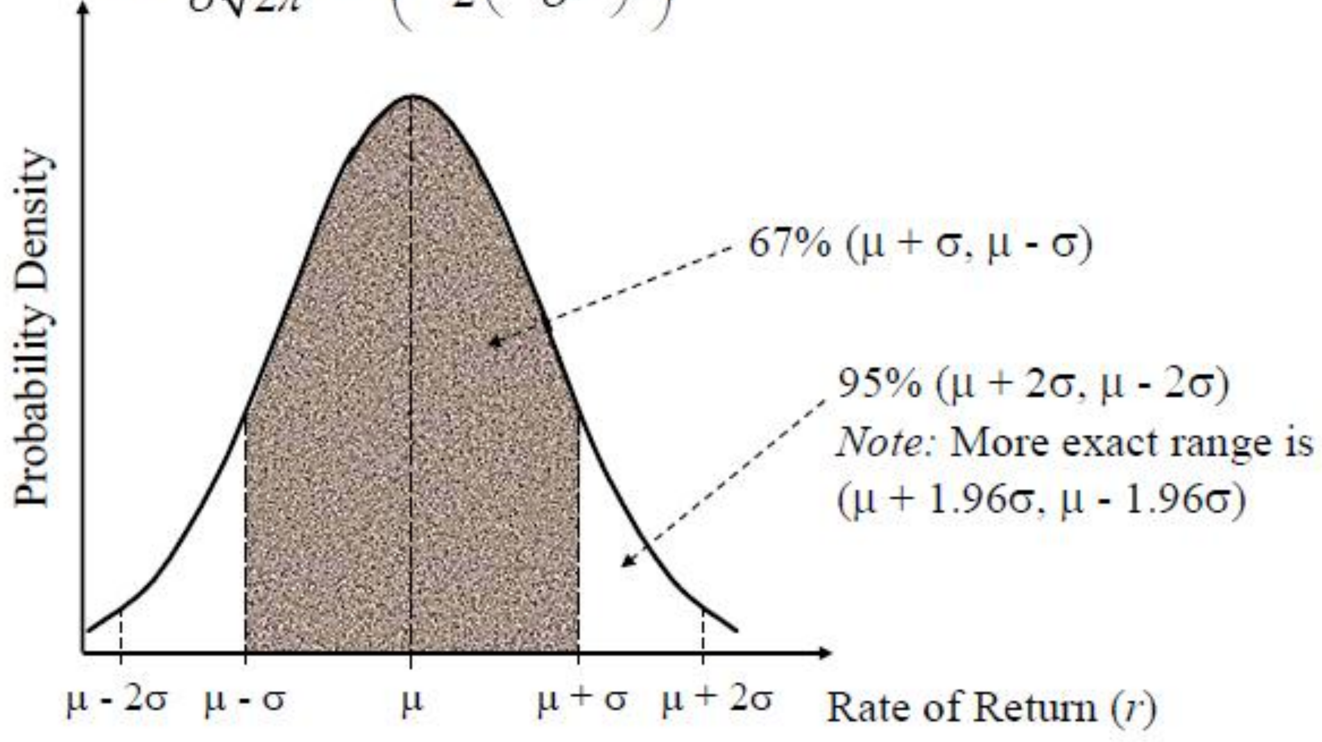
  - Only the expected (or mean) return and standard deviation of return are needed to fully describe the distribution of returns

$$f(r) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left( -\frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 \right)$$

Note: A normal distribution implies an unlimited downside loss potential - unrealistic assumption because the most you can lose on a security purchase is 100%!

# Interpreting Return and Risk Measures

$$f(r)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2\right)$$



# Interpreting Return and Risk Measures

Given $\mu_X$ = 7% and $\sigma_X$ = 11.9%

Assume you purchased 100 shares of Stock X, investing $1,000

- The expected value of this security next period is $1000(1+\mu) = 1000(1.07) = \$1,070$
  - There is a 67% probability that the realized return will lie in the range ($\mu - \sigma$, $\mu + \sigma$) or (-4.9%, 18.9%)
  - There is a 95% probability that the realized return will lie in the range ($\mu - 2\sigma$, $\mu + 2\sigma$) or (-16.8%, 30.8%)

There is a 95% probability that a $1000 investment in this security will be worth between 1000(1 - 0.168) = $832 and 1000(1 + 0.308) = $1,308 next period

The range of final wealth is $832 - $1,308 with a 95% likelihood of the actual/realized wealth lying in this range

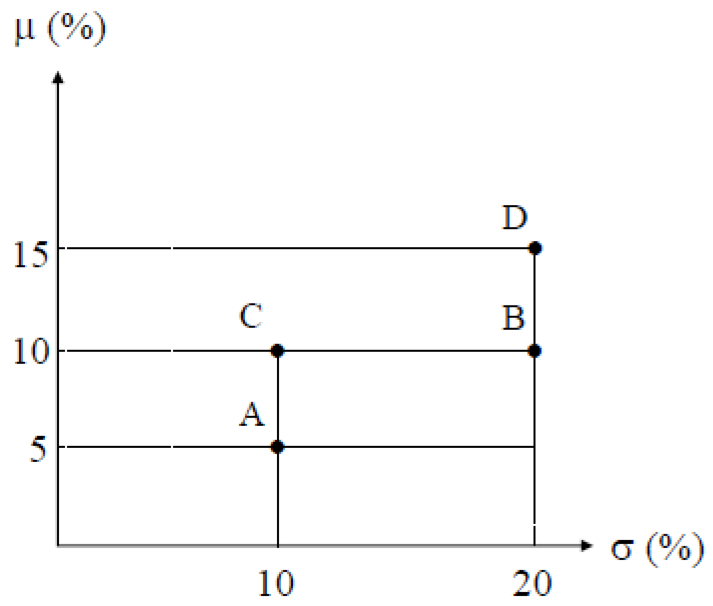
# Specifying Investor Preferences

You're given the following information on four risky securities. Which securities would a risk averse investor prefer?

| Security | Expected Return | Standard Deviation of Returns |
|---|---|---|
| A | 5% | 10% |
| B | 10% | 20% |
| C | 10% | 10% |
| D | 15% | 20% |

• Risk averse investors would prefer the highest expected returns and the lowest risk levels
• C is preferred to A
• C is preferred to B
• D is preferred to B

• What about A and B, C and D, A and D?



# Specifying Investor Preferences

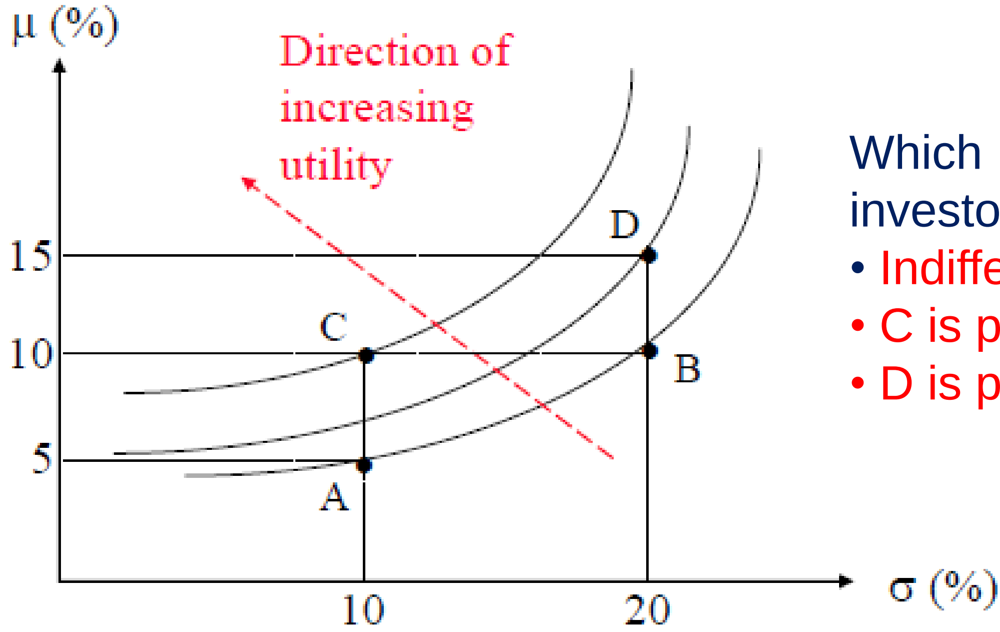
Assuming asset returns are normally distributed, investor utility can be written as a function of $\mu$ and $\sigma^2$: $U = f(\mu, \sigma^2)$

Example:

$U = \mu - 0.3\sigma^2$ (this is just an example of a utility function)

$\Delta U/\Delta\mu > 0$ (Higher expected return preferred to lower expected return)

$\Delta U/\Delta\sigma^2 < 0$ (Lower risk preferred to higher risk)

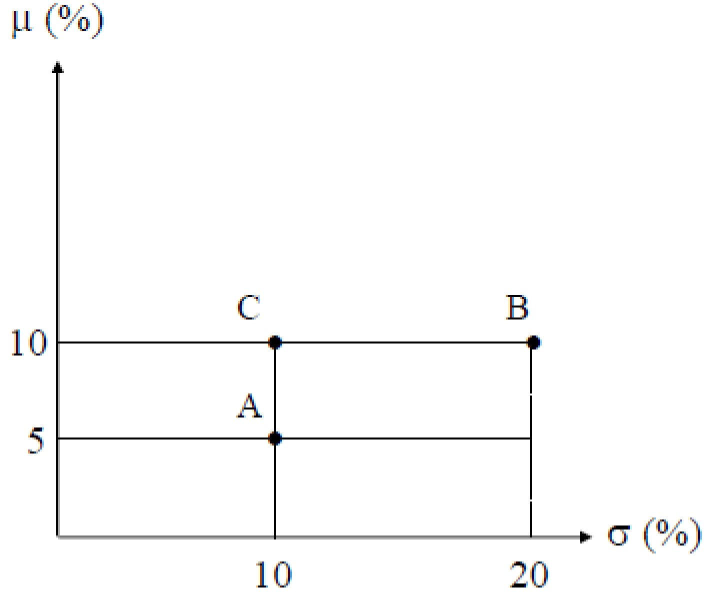



Which securities would a risk averse investor now prefer?

- Indifferent between A and B
- C is preferred to A, B and D
- D is preferred to A and B

# Specifying Investor Preferences

- Example: Evaluate securities A, B and C based on their coefficients of variation




- CV = σ / μ
  - $CV_A = 0.10/0.05 = 2.0$
  - $CV_B = 0.20/0.10 = 2.0$
  - $CV_C = 0.10/0.10 = 1.0$

- C is preferred to A and B (as before)
- Investor is indifferent between A and B - same risk per unit expected return

# Portfolios and Risk Diversification

A risk averse investor's objective is to

- Minimize the risk of portfolio of investments, given a desired level of expected return, or
- Maximize the expected return of portfolio of investments, given a desired level of risk

The easiest way to minimize risk is to diversify across different assets by forming a portfolio (collection) of assets

- Portfolio risk falls as the number of assets in the portfolio Increases
- Portfolio risk cannot be entirely eliminated using this method
  - The risk not eliminated is called systematic risk

# Portfolio Risk and Return: Two Assets

Portfolio's expected return is the weighted average of the expected returns of its component assets - weights are the percentage of investor's original wealth invested in each asset

$$E(R_P) = \mu_p = w_1 E(R_1) + w_2 E(R_2)$$

$w_j$ = Amount invested in asset j / Total invested in all assets
Note: $w_1 + w_2 = 1$ and $w_1 = 1 - w_2$ (or $w_2 = 1 - w_1$)

Portfolio's variance is the weighted average of the variance of its component assets and the covariance between the assets' returns

$$Var(R_P) = \sigma_P^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 Cov(r_1, r_2)$$

# Covariance Between Asset Returns

Covariance measures the level of co-movement between assets

$$Cov_{1,2} = p_1(r_{11} - \mu_1)(r_{21} - \mu_2) + \cdots p_n(r_{1n} - \mu_1)(r_{2n} - \mu_2)$$

$r_{jk}$ = Return on asset j = 1, 2 in state k = 1, 2, …, n

$\sigma_{12} > 0$: Above (below) average returns on asset 1 tend to occur with above (below) average returns on asset 2

$\sigma_{12} < 0$: Above (below) average returns on asset 1 tend to occur with below (above) average returns on asset 2

The magnitude of covariance may change depending on how returns are measured

- Dollars versus cents
- Percentages versus decimals

# Correlation Between Asset Returns

Correlation coefficient is a "standardized" measure of co-movement between two assets

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}, \text{where } -1 \leq \rho_{1,2} \leq +1$$

Note: The sign of the correlation coefficient is the same as the sign of the return covariance

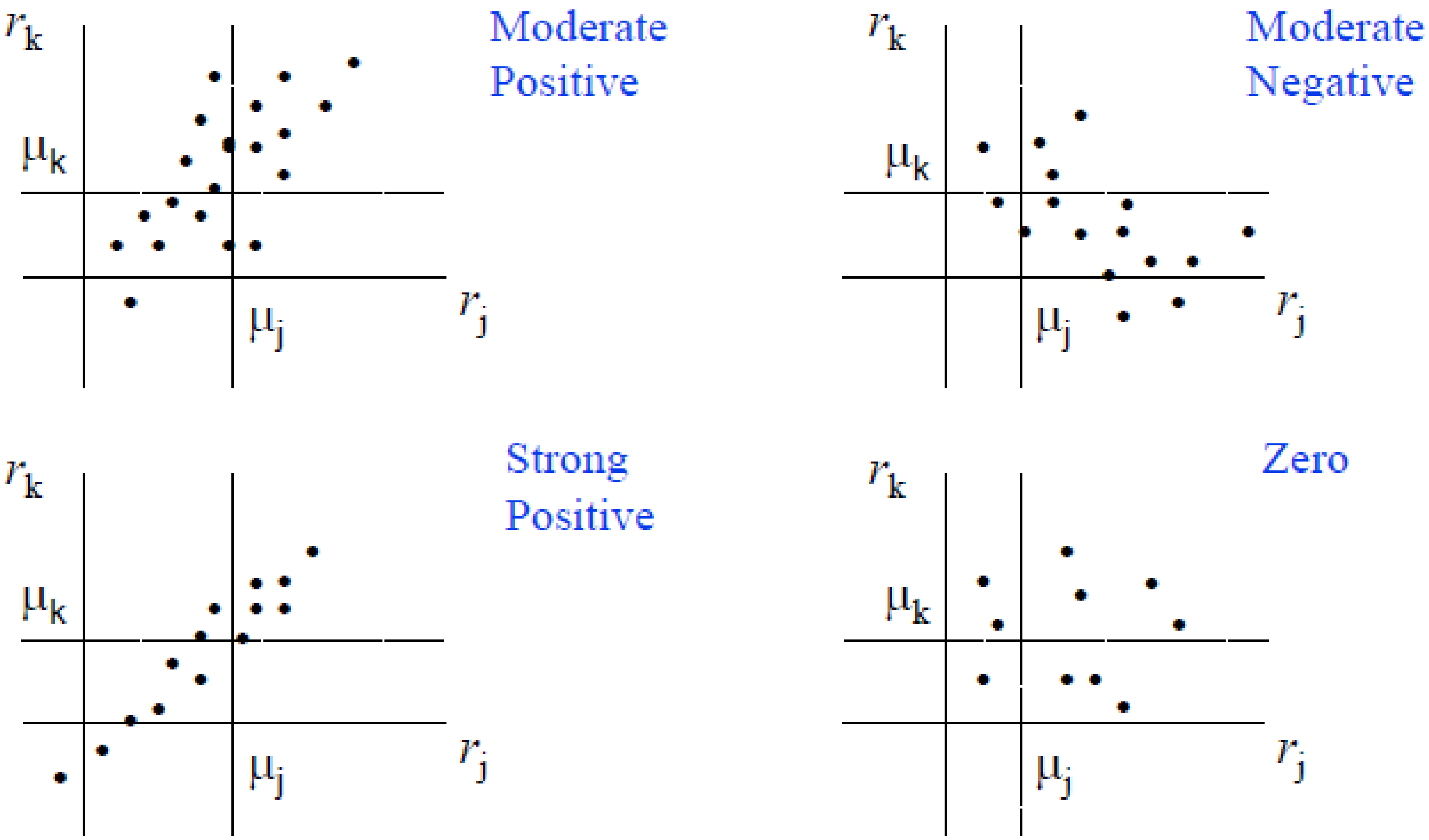
The covariance of returns can be rewritten as

$$\sigma_{12} = \sigma_1 \sigma_2 \rho_{12}$$

A portfolio's variance can be rewritten as the weighted average of the variance of its component assets and the correlation between the assets' returns

$$Var(R_P) = \sigma_P^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_1 \sigma_2 \rho_{12}$$

# Examples of Asset Correlations



# Return Correlations of Selected Stocks: 1993-2003

| | AOI | ANZ | BHP | CBA | CML | FGL | NAB | SRP |
|---|---|---|---|---|---|---|---|---|
| AOI | 1.00 | 0.62 | 0.61 | 0.59 | 0.33 | 0.41 | 0.62 | 0.39 |
| ANZ | | 1.00 | 0.29 | 0.68 | 0.22 | 0.27 | 0.61 | 0.38 |
| BHP | | | 1.00 | 0.22 | 0.12 | 0.16 | 0.34 | 0.28 |
| CBA | | | | 1.00 | 0.28 | 0.25 | 0.69 | 0.28 |
| CML | | | | | 1.00 | 0.11 | 0.22 | 0.30 |
| FGL | | | | | | 1.00 | 0.26 | 0.25 |
| NAB | | | | | | | 1.00 | 0.30 |
| SRP | | | | | | | | 1.00 |
| Mean | 6.5% | 17.6% | 4.8% | 15.3% | 4.3% | 5.9% | 14.4% | -0.9% |
| Median | 9.5% | 16.5% | 0.0% | 14.0% | -1.7% | 0.0% | 22.0% | -8.5% |
| Std Dev | 12.9% | 21.3% | 22.8% | 19.8% | 20.4% | 17.3% | 21.2% | 28.7% |

*Note:* The sample period is Jan 1993 - Jun 2003. AOI refers to the All Ordinaries Index which is a proxy for the "market" portfolio. Stocks tend to be positively correlated with each other and with the market portfolio. The mean, median and standard deviation of returns are based on continuously compounded monthly returns which have been annualized

# Example

$$Suppose\ \sigma_1^2 = 5,\ \sigma_2^2 = 8\ and\ \sigma_{1,2} = -3$$

Calculate the Variance of the portfolios:
10% in asset 1 and 90% in asset 2
80% in asset 1 and 20% in asset 2
40% in asset 1 and 60% in asset 2

$$\sigma_P^2 = 0.1^2\sigma_1^2 + 0.9^2\sigma_2^2 + 2 \times 0.1 \times 0.9\sigma_{1,2} = 5.99$$
$$\sigma_P^2 = 0.8^2\sigma_1^2 + 0.2^2\sigma_2^2 + 2 \times 0.8 \times 0.2\sigma_{1,2} = 4.36$$
$$\sigma_P^2 = 0.4^2\sigma_1^2 + 0.6^2\sigma_2^2 + 2 \times 0.4 \times 0.6\sigma_{1,2} = 2.24$$

Second and third portfolios have produced variances that are smaller than the variances of the individual assets.

**Example of Diversification Effect**

# Portfolios and Risk Diversification

## Key implications

As long as individual securities are not perfectly positively correlated, diversification benefits will always exist in the form of reduction of portfolio risk

Diversification benefits can be maximized if securities are negatively correlated with each other

•As the number of securities in a portfolio increases, the covariance of returns between different securities determines the portfolio's total risk.

## Zero correlation case:

Combining the two securities produces a portfolio variance, which is half of the variance of the assets.

## Perfect negatively correlated case:

All risk is eliminated, the portfolio variance becomes zero

## Perfect positively correlated case:

Combining the assets does not reduce risk

## Case Study

The average return and standard deviation of returns for 14 FTSE 100 companies in the period 2000-2005 quoted on the stock exchange are given below:

| Company name | Average Monthly Return | Monthly Standard Deviation | Company name | Average Monthly Return | Monthly Standard Deviation |
|---|---|---|---|---|---|
| ASTRAZENECA | -0.47% | 8.43% | CADBURY SCHWEPPES | 0.43% | 5.87% |
| BAA | 0.49% | 6.50% | DIXONS GP. | -1.50% | 11.58% |
| BOOTS GROUP | 0.14% | 7.03% | HILTON GROUP | 0.60% | 9.93% |
| BP | -0.34% | 6.17% | LAND SECURITIES | 1.18% | 5.13% |
| BRIT.AMERICAN TOBACCO | 1.56% | 8.20% | MORRISON(WM)SPMKTS. | 0.74% | 7.57% |
| BRIT.SKY BCAST. | -0.95% | 11.69% | NATIONAL GRID TRANSCO | 0.09% | 5.90% |
| CABLE & WIRELESS | -3.62% | 17.58% | RIO TINTO | 0.04% | 9.00% |

The correlation coefficients for all pairs of asset returns is given in the next matrix

| | ASTRAZEI | BAA | BOOTS GI | BP | BRIT.AMEI | BRIT.SKY | CABLE & \ | CADBURY | DIXONS G | HILTON GI | LAND SEC | MORRISO | NATIONAL | RIO TINTO |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ASTRAZENECA | 1 | | | | | | | | | | | | | |
| BAA | 0.333089 | 1 | | | | | | | | | | | | |
| BOOTS GROUP | 0.441573 | 0.507357 | 1 | | | | | | | | | | | |
| BP | 0.24099 | 0.254502 | 0.308227 | 1 | | | | | | | | | | |
| BRIT.AMERICAN TOBACCO | 0.282825 | 0.328933 | 0.422053 | 0.538362 | 1 | | | | | | | | | |
| BRIT.SKY BCAST. | -0.084352 | -0.010685 | -0.056557 | 0.025517 | -0.085005 | 1 | | | | | | | | |
| CABLE & WIRELESS | -0.059292 | 0.1067 | 0.013345 | -0.024106 | 0.123683 | 0.338221 | 1 | | | | | | | |
| CADBURY SCHWEPPES | 0.296985 | 0.470908 | 0.441398 | 0.210766 | 0.293434 | 0.053581 | -0.029348 | 1 | | | | | | |
| DIXONS GP. | 0.163226 | 0.361705 | 0.397972 | 0.085731 | -0.091778 | 0.112503 | 0.132865 | 0.40952 | 1 | | | | | |
| HILTON GROUP | 0.112812 | 0.236032 | 0.34243 | 0.325337 | 0.202883 | 0.376362 | 0.244006 | 0.419682 | 0.383177 | 1 | | | | |
| LAND SECURITIES | 0.251717 | 0.265184 | 0.464362 | 0.375558 | 0.36956 | 0.242802 | 0.142483 | 0.413168 | 0.272424 | 0.438767 | 1 | | | |
| MORRISON(WM)SPMKTS. | 0.250837 | 0.211243 | 0.237251 | 0.128989 | 0.259888 | -0.051157 | -0.11411 | 0.45449 | 0.247104 | -0.058254 | 0.21702 | 1 | | |
| NATIONAL GRID TRANSCO | 0.100491 | 0.200382 | 0.325697 | 0.249417 | 0.355011 | 0.082219 | -0.01142 | 0.353064 | 0.293258 | 0.405293 | 0.402676 | 0.204109 | 1 | |
| RIO TINTO | 0.205672 | 0.329183 | 0.403845 | 0.523512 | 0.391198 | 0.127054 | 0.096968 | 0.150319 | 0.118554 | 0.365247 | 0.266138 | 0.071659 | 0.14095 | 1 |

An equally weighted portfolio will produce:

Expected return: =10.12% and Risk (Standard Deviation) = 4.40%

# Portfolio Risk and Return: Many Assets

The expected return of an M asset portfolio is

$$E(r_p) = \sum_{j=1}^{m} w_j E(r_j)$$

The standard deviation of an M asset portfolio is

$$\sigma_p = \sqrt{\sum_{j=1}^{m}\sum_{k=1}^{m} w_j w_k \sigma_{jk}}$$

*Note:* $\sum_{j=1}^{m}$ is the summation operator - Sum all elements after the operator from $j = 1$ to $j = \mathrm{m}$

$\sum_{j=1}^{m}\sum_{k=1}^{m}$ is a double summation operator - For $j = 1$ sum all the elements from $k = 1$ to $k = \mathrm{m}$, for $\mathrm{j} = 2$ sum all the elements from $k = 1$ to $k = \mathrm{m}$, and so on until $j = \mathrm{m}$

# Portfolio Risk and Return: Many Assets

$$\sigma_p = \sqrt{\sum_{j=1}^{m}\sum_{k=1}^{m} w_j w_k \sigma_{jk}}$$ = Square root of sum of all elements in table

$m = 2$

$$\begin{array}{l} \\ \text{Asset 1} \\ \text{Asset 2} \\ \text{Asset 3} \\ \text{Asset 4} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Asset 1} & \text{Asset 2} & \text{Asset 3} & \text{Asset 4} \end{array} \\ \begin{bmatrix} w_1 w_1 \sigma_{11} & w_1 w_2 \sigma_{12} & w_1 w_3 \sigma_{13} & w_1 w_4 \sigma_{14} \\ w_2 w_1 \sigma_{21} & w_2 w_2 \sigma_{22} & w_2 w_3 \sigma_{23} & w_2 w_4 \sigma_{24} \\ w_3 w_1 \sigma_{31} & w_3 w_2 \sigma_{32} & w_3 w_3 \sigma_{33} & w_3 w_4 \sigma_{34} \\ w_4 w_1 \sigma_{41} & w_4 w_2 \sigma_{42} & w_4 w_3 \sigma_{43} & w_4 w_4 \sigma_{44} \end{bmatrix} \end{array} \quad m = 4$$

- Total number of SD terms = $m$ = 4; Covariance terms = $m(m-1)$ = 12
  (*Note:* When $m$ = 100, SD terms = 100; Covariance terms = 9,900!)
- $\sigma_{jk} = \rho_{jk}\, \sigma_j\, \sigma_k$
- $\rho_{jk}$ = Correlation coefficient between returns for securities $j$ and $k$
- $\sigma_j$ , $\sigma_k$ = Standard deviations for security $j$ and $k$
- When $j = k$: $\rho_{jk} = 1$ and $\sigma_{jj} = \sigma_j^2$ (see diagonal terms)

# Example: Three Risky Assets

| | | Covariance Matrix | | |
|---|---|---|---|---|
| | E (ri) | Apple | Sun Micro | Red Hat |
| Apple | 0.20 | 0.09 | 0.045 | 0.05 |
| Sun Micro | 0.12 | 0.045 | 0.07 | 0.04 |
| Red Hat | 0.15 | 0.05 | 0.04 | 0.06 |

The diagonal of the Covariance Matrix holds Variances

$$E(R_P) = w_A E(R_A) + w_S E(R_S) + w_H E(R_H) = \frac{1}{3} \times 0.2 + \frac{1}{3} \times 0.12 + \frac{1}{3} \times 0.15 = 15.7\%$$

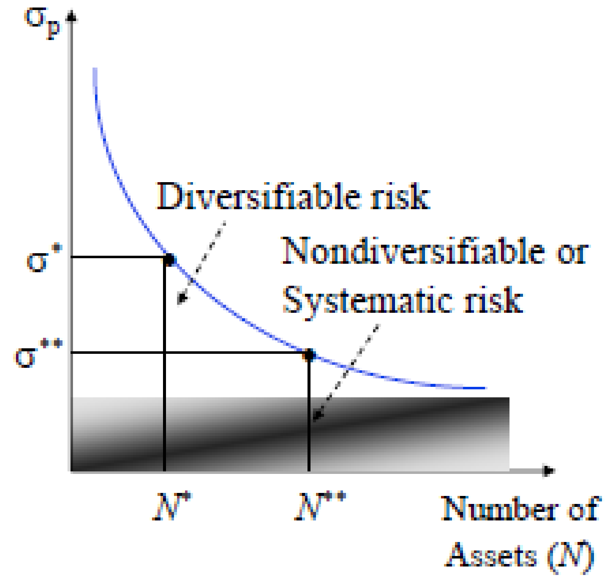
$$\sigma^2(R_P) = w_A^2 \sigma^2(R_A) + w_S^2 \sigma^2(R_S) + w_H^2 \sigma^2(R_H) +$$
$$+2 w_A w_S Cov(R_A, R_S) + 2 w_A w_H Cov(R_A, R_H) + 2 w_S w_H Cov(R_S, R_H)$$

$$\sigma^2(R_P) = \frac{1}{9}(0.09) + \frac{1}{9}(0.07) + \frac{1}{9}(0.06) + 2\left(\frac{1}{3}\frac{1}{3}\right)(0.045) + 2\left(\frac{1}{3}\frac{1}{3}\right)(0.05) + 2\left(\frac{1}{3}\frac{1}{3}\right)(0.04)$$

$$= 0.0544 \; and \; \sigma(R_P) = \sqrt{0.0544} = 23.3\%$$

# Limits to Diversification Benefits

As the number of securities in a portfolio increases, the risk of the portfolio decreases




Assume: $w_j = 1/N$, $\sigma_j = \sigma_k = \sigma$,
$\rho_{jk} = \rho < +1$ for all j and k

Only in this case, we have:

$$\sigma_p^2 = \sigma^2/N + [(N - 1)/N]\sigma_{jk}$$

As N increases, the first term approaches zero and the second term approaches $\sigma_{jk}$

Implication:

The risk of large portfolios is determined by asset covariance

# Minimizing The Portfolio Variance

A small thought experiment

To illustrate the importance of covariances lets think about how we could find the minimum variance portfolio.

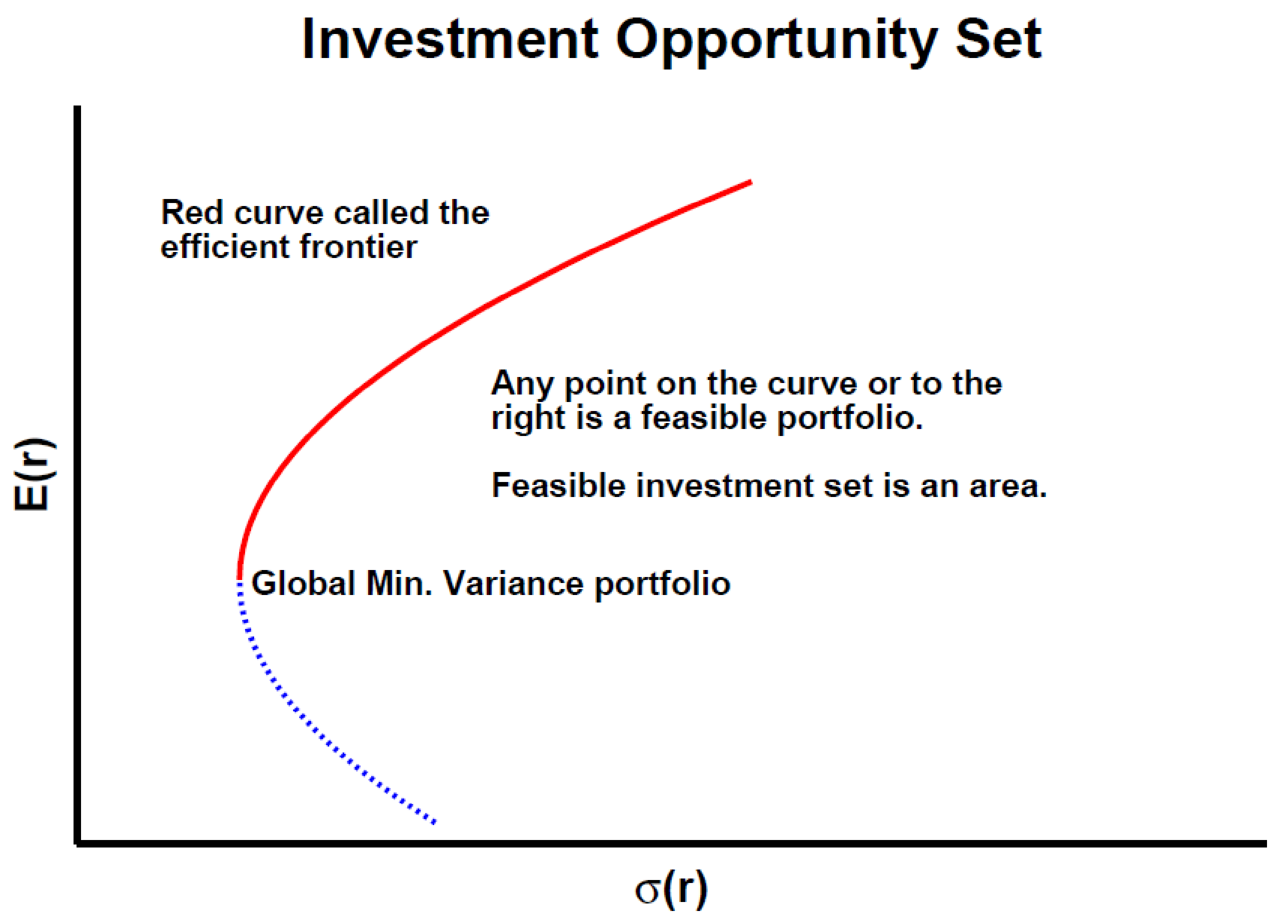
Finding the portfolio with the smallest possible variance

- Find two securities already in the portfolio with different covariances with the portfolio
- Add a little weight to the security with a lower *cov* $(r_i, r_p)$ and subtract a little from the security with the higher covariance.
- The portfolio variance is a little lower. Repeat steps 1 and 2 until the variance cannot be lowered anymore.

The variance of the portfolio will be minimized when all the securities have the same covariance with the portfolio

$$Cov(R_{1,}R_P) = Cov(R_{2,}R_P) = \cdots = Cov(R_{n,}R_P)$$

# Minimum Variance frontier and Efficient Frontier of N-Assets



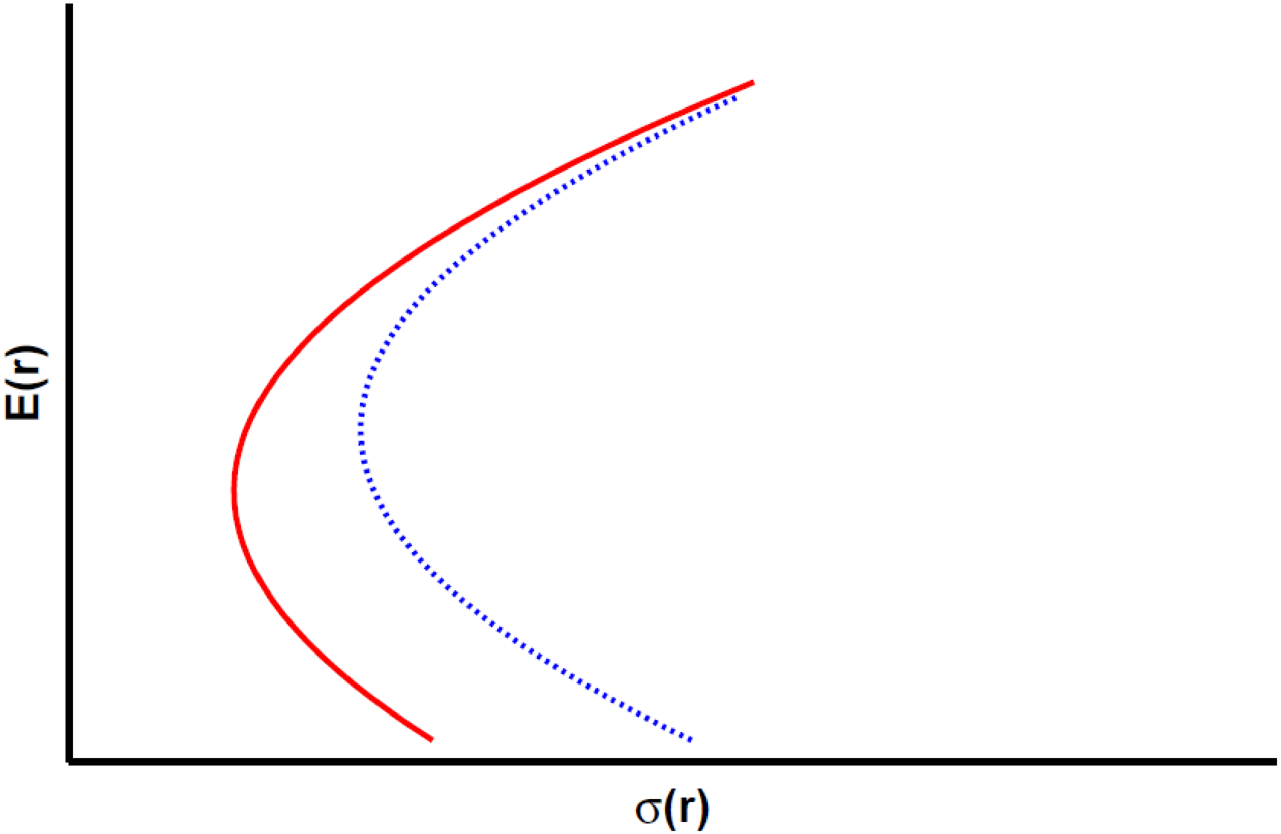
# The Investment Opportunity Set

When there are many risky assets the feasible investment set is a curve and the area to the right of the curve

The frontier

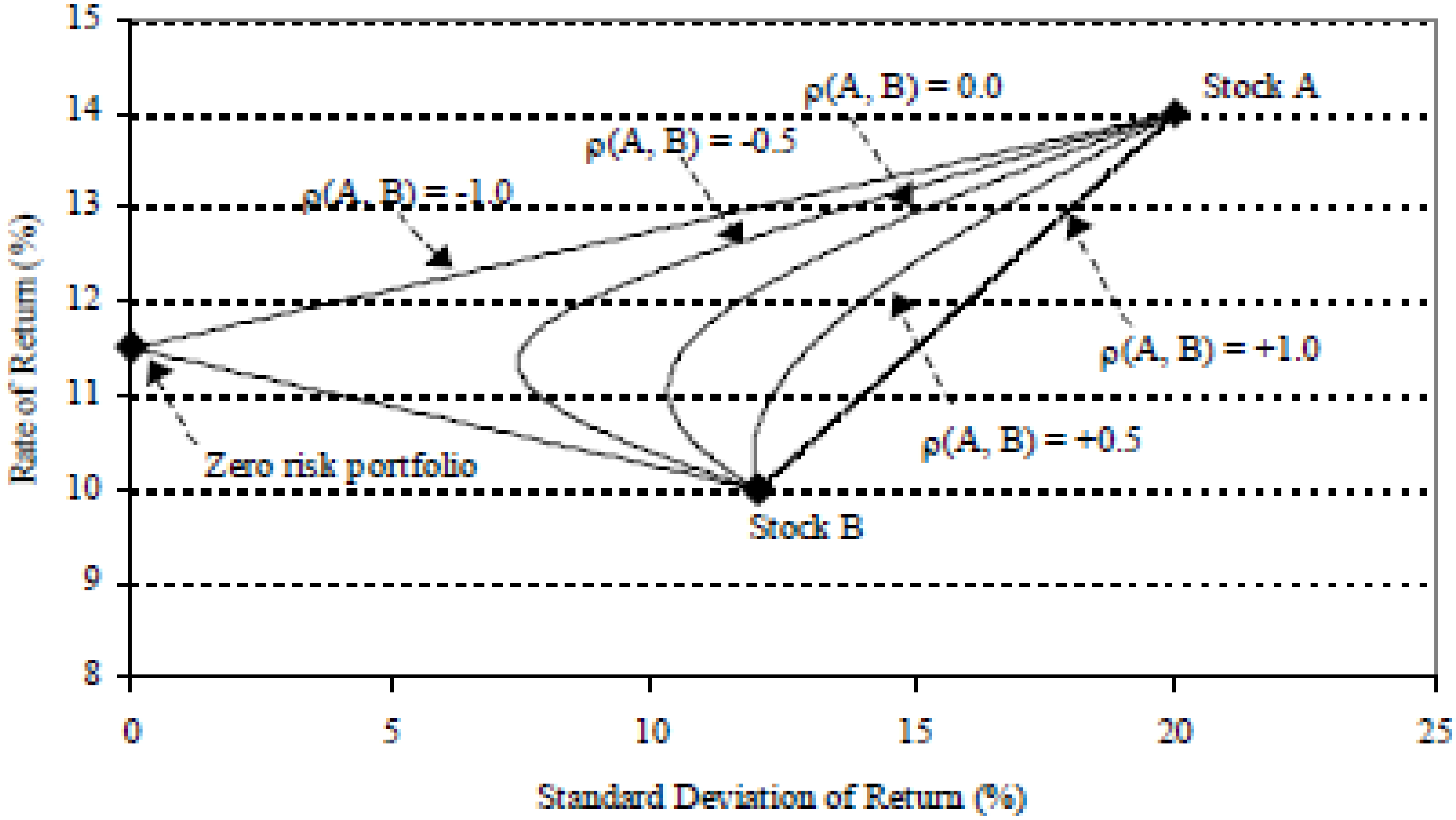
- The curve in the graph is called the mean-variance boundary or the mean variance frontier
- The portion of the curve above the global minimum variance portfolio is called the mean-variance efficient frontier or just the efficient frontier
- An investor will only choose a portfolio on the efficient frontier
- Portfolios on the efficient frontier are called mean variance efficient portfolios or efficient portfolios
- Adding more assets pushes the curve towards the northwest corner of the graph

# Mean Variance Frontier: Adding Assets



# Summary of Risk-Return Tradeoffs

Risk and expected return tradeoffs for different levels of correlation between returns



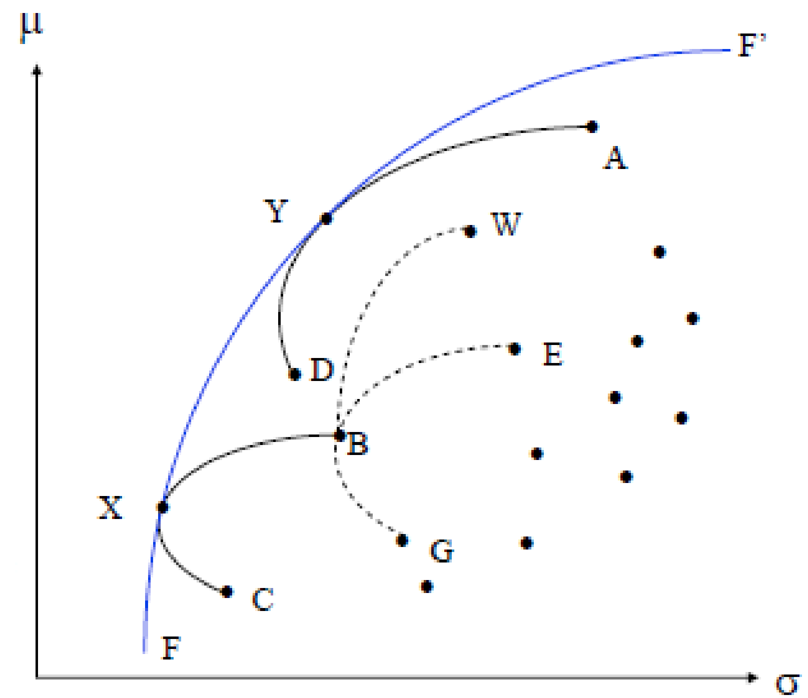
# The efficient frontier of risky assets and individual assets

The efficient frontier FF' is an envelope of individual risk-return Frontiers

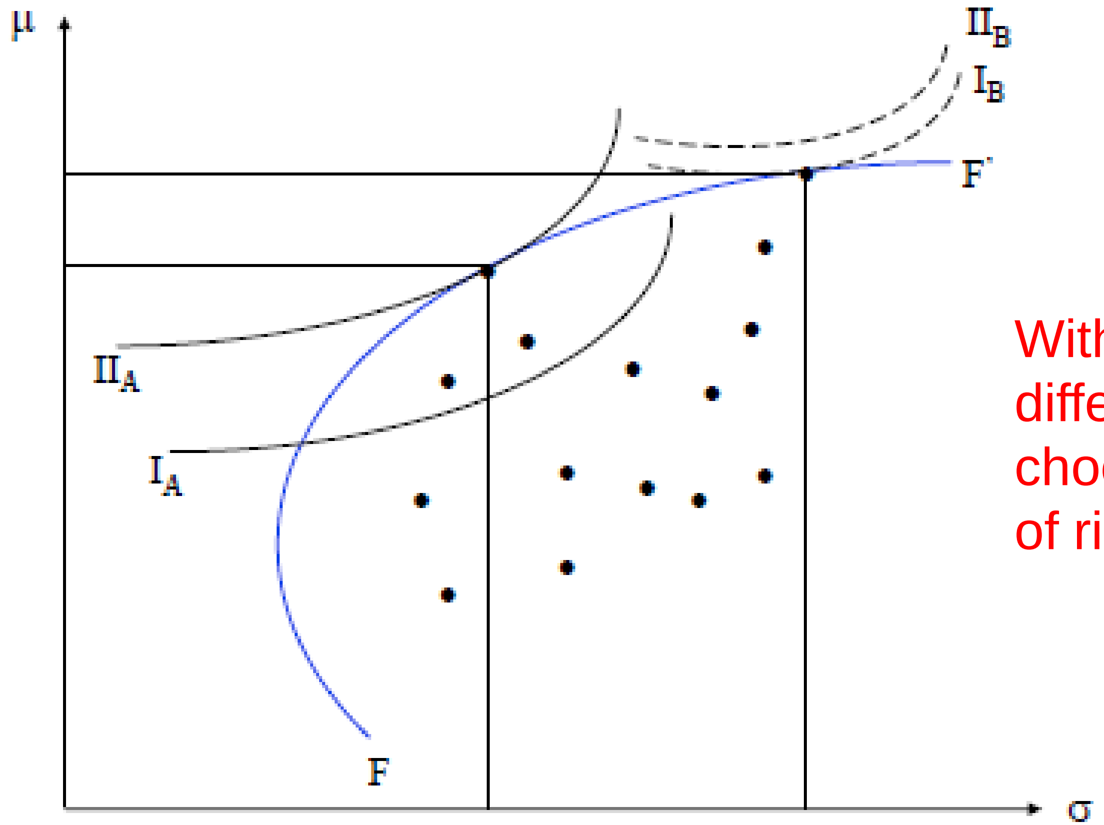
X is a portfolio of portfolios B and C and Y is a portfolio of A and D

FF' plots risky portfolios which have the lowest risk (σ) for a given expected return (μ)

Note: Portfolio W dominates E (higher m; lower s), but E may still be included in a portfolio because of low (or negative) covariance effects



# Investor Choice Without a Riskfree Asset




Without a riskfree asset different investors will choose different portfolios of risky assets

# The Efficient Frontier with Short Sales Allowed

Constraints may preclude a particular investor from choosing portfolios on the efficient frontier.

Possible constraint: Short sale (usual regulated type of market transaction)

Involves selling assets that are borrowed in expectation of a fall in the assets' price. The investor buys an equivalent number of assets at the new lower price and returns to the lender the assets that was borrowed.

If,

$$E(r_A) \geq E(r_B) \quad \text{and} \quad \sigma_A \geq \sigma_B$$

Investors could sell the lowest-return asset B.

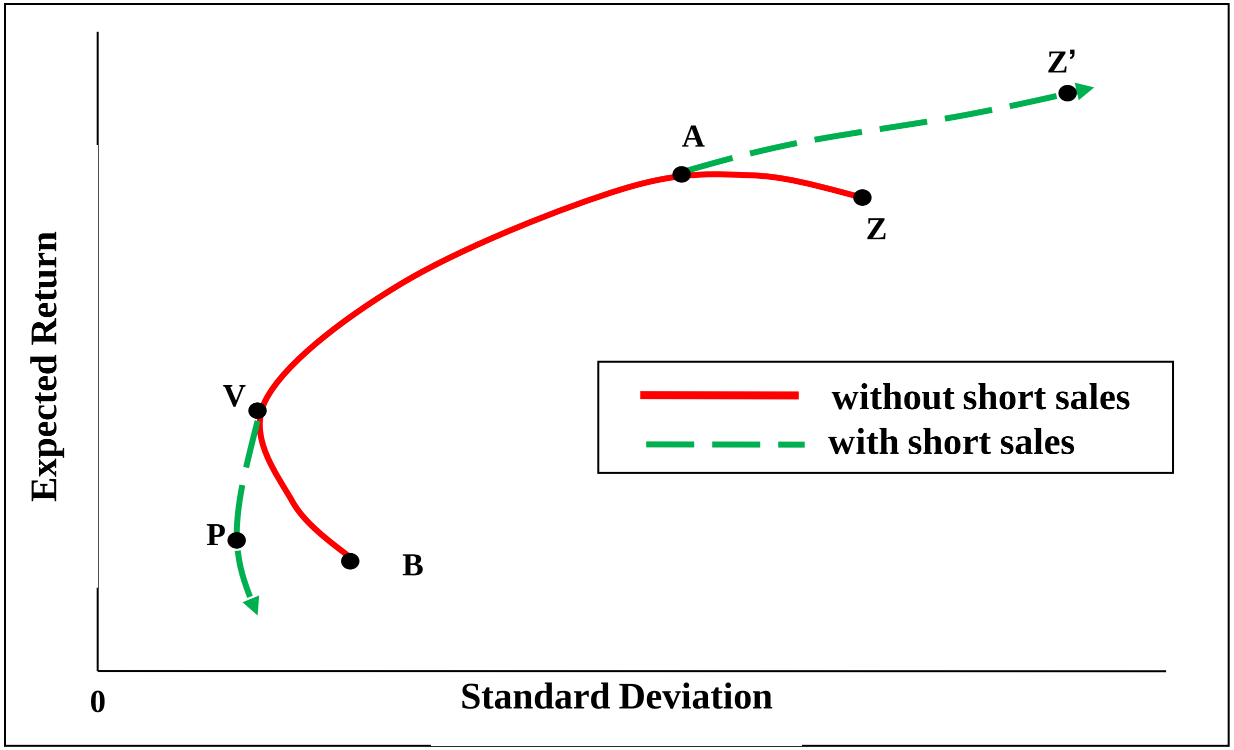
If the number of short sales is unrestricted, then by a continuous short selling of B and reinvesting in A the investor could generate an infinite expected return.

The upper bound of the highest-return portfolio would no longer be A but infinity

Investor could short sell the highest-return security A and reinvest the proceeds into the lowest-yield security B

In this case will generate a return less than the return on the lowest-return assets. Rational investor will not short sell a high-return asset and buy a low-return asset. This case is just for extreme assumption.

# The Efficient Frontier with Short Sales Allowed



# Portfolios of one Risky and one Risk-Free Asset

Inclusion of a risk-free asset will simplify the analyse.

Risk free asset: rate of return over the holding period is certain

No uncertainty, therefore by definition the standard deviation is zero.

Covariance among rate of return of a risk free asset and any risky asset is zero.

Riskless asset: fixed income security with no possibility of default.

Two types of investments: *lending* (buying short term Government bonds) or *borrowing* (borrowing a risk free security in order to short sell and use the proceeds for investment in risky assets).

## Let's assume:

Risk Free Rate = $r_F$ and risky asset = $r_E$, and $W_E + W_F = 1$

Therefore, the expected return of a risky asset-bond portfolio

$$E(R_P) = W_E E(R_E) + (1 - W_E) R_F$$

The variance of such a portfolio is then:

$$\sigma_P^2 = w_E^2 \sigma_E^2 + (1 - w_E)^2 \sigma_F^2 + 2 w_E (1 - w_E) \sigma_{E,F}$$

Since for the risk-free asset

$$\sigma_F = 0 \qquad and \qquad \sigma_{E,F} = \rho \sigma_E \sigma_F = 0$$

The expected return and the variance of the risky asset-bond portfolio can be written as:

$$E(R_P) = R_F + w_E[E(R_E) - R_F], \text{and} \qquad \sigma_P^2 = w_E^2 \sigma_E^2$$

The standard deviation of a resulting portfolio is then:

$$\sigma_P = w_E \sigma_E$$

and finally,

$$E(R_P) = R_F + \frac{E(R_E) - R_F}{\sigma_E} \sigma_P$$

## Example

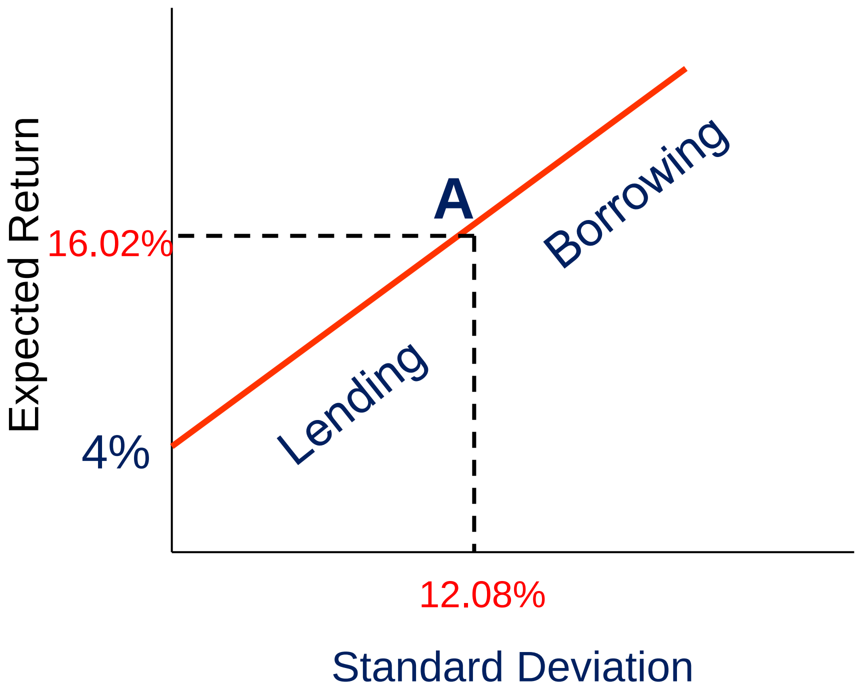
Three stocks with the following characteristics:

| Stock | Expected Return (%) | Covariance | | |
|---|---|---|---|---|
| | | A | B | C |
| A | 16.2 | 146 | 187 | 145 |
| B | 24.6 | 187 | 854 | 104 |
| C | 22.8 | 145 | 104 | 289 |

Risk free asset with a rate of return of 4% and variance of zero.

Consider the following risk-return scenarios

| Asset A | Risk Free Asset | Expected Return (%) | Standard Deviation (%) |
|---|---|---|---|
| 0 | 1 | 4.00 | 0.00 |
| 0.25 | 0.75 | 7.05 | 3.02 |
| 0.50 | 0.50 | 10.10 | 6.04 |
| 0.75 | 0.25 | 13.15 | 9.06 |
| 1.00 | 0 | 16.20 | 12.08 |
| 1.25 | -0.25 | 19.25 | 15.10 |
| 1.50 | -0.50 | 22.30 | 18.12 |
| 1.75 | -0.75 | 25.35 | 21.14 |
| 2.00 | -1.00 | 28.40 | 24.16 |



# Portfolios of two Risky assets and one Risk-Free Asset

Expected return of 3-asset portfolio:

$$E(r_p) = w_{E1}E(r_{E1}) + w_{E2}E(r_{E2}) + w_f r_f$$

Variance of the portfolio:

$$\sigma_p^2 = w_{E1}^2\sigma_{E1}^2 + w_{E2}{}^2\sigma_{E2}^2 + w_f^2\sigma_f^2 + 2w_{E1}w_{E2}\sigma_{E1,E2} + 2w_{E1}w_f\sigma_{E1,f} + 2w_{E2}w_f\sigma_{E2,f}$$

Since: $w_{E1} + w_{E2} + w_f = 1$

$$E(r_p) = r_f + w_{E1}[E(r_{E1}) - r_f] + w_{E2}[E(r_{E2}) - r_f]$$

*and*

$$\sigma_p^2 = w_{E1}^2\sigma_{E1}^2 + w_{E2}^2\sigma_{E2}^2 + 2w_{E1}w_{E2}\sigma_{E1,E2}$$

Written in the matrix form,

$$E(r_p) = r_f + \begin{bmatrix} w_{E1} & w_{E2} \end{bmatrix} \begin{bmatrix} E(r_{E1}) - r_f \\ E(r_{E2}) - r_f \end{bmatrix} = r_f + \mathbf{w}'(\mathbf{r} - \mathbf{1} r_f) = \mathbf{w}'\mathbf{r} + (1 - \mathbf{w}'\mathbf{1}) r_f$$

Where:

$$\mathbf{w} = \begin{bmatrix} w_{E1} \\ w_{E2} \end{bmatrix} \; and \quad \mathbf{r} = \begin{bmatrix} E(r_{E1}) \\ E(r_{E2}) \end{bmatrix}$$

Similarly, the variance of the two risky asset-bond portfolios can be written in the matrix form:

$$\sigma_p^2 = \begin{bmatrix} w_{E1} & w_{E2} \end{bmatrix} \begin{bmatrix} \sigma_{E1}^2 & \sigma_{E1,E2} \\ \sigma_{E2,E1} & \sigma_{E2}^2 \end{bmatrix} \begin{bmatrix} w_{E1} \\ w_{E2} \end{bmatrix} = \mathbf{w}'\Sigma\mathbf{w}$$

*where*

$$\Sigma = \begin{bmatrix} \sigma_{E1}^2 & \sigma_{E1,E2} \\ \sigma_{E2,E1} & \sigma_{E2}^2 \end{bmatrix} \text{ is the variance-covariance matrix}$$

## Example

Three stocks with the following characteristics:

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| C | 22.8 | 145 | 104 | 289 |

Consider an investor who wants to create a portfolio from risky assets A and C and a risk free asset. Assume investor wants to invest 75% in risk free asset and 25% in risky assets A and C. Furthermore he wants to invest 80% in asset A and 20% in asset C.

## Portfolio Return

$$r_p = w_R\left(w_{E1R}r_{E1} + w_{E2R}r_{E2}\right) + w_f r_f = w_R r_R + w_f r_f$$
$$= 0.25\times(0.80\times 0.162 + 0.20\times 0.228) + 0.75\times 0.04 = 7.38\%$$

## Standard Deviation of a risky portfolio R

$$\sigma_R = \sqrt{0.80^2\times 146 + 0.20^2\times 189 + 2\times 0.80\times 0.20\times 145} = 12.30\%$$

## Standard Deviation of a a portfolio that has part of funds invested in AC and part in riskless is:

$$\sigma_p = w_R\sigma_R = 0.25\times 0.1230 = 3.08\%$$

## Case with risk-free borrowing

Suppose the investor borrow 25% and invest 125% in the risky portfolio

$$r_p = 20.90\% \quad and \quad \sigma_p = 15.38\%$$

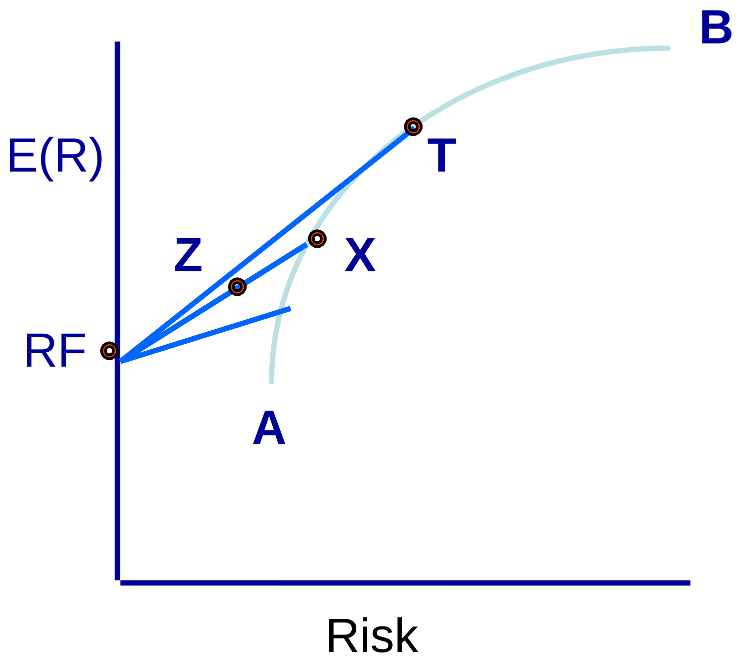
# Riskless Lending and Borrowing and Efficient Frontiers

## Risk-free assets

- Certain-to-be-earned expected return, zero variance
- No correlation with risky assets
- Usually proxied by a Treasury Bill
  - Amount to be received at maturity is free of default risk, known with certainty

Adding a risk-free asset extends and changes the efficient frontier

# Risk-Free Lending




- Riskless assets can be combined with any portfolio in the efficient set AB
  - Z implies lending

- The efficient frontier is the straight line RF – T and the remaining part of the original, concave efficient frontier

# Borrowing Possibilities

Investor no longer restricted to own wealth

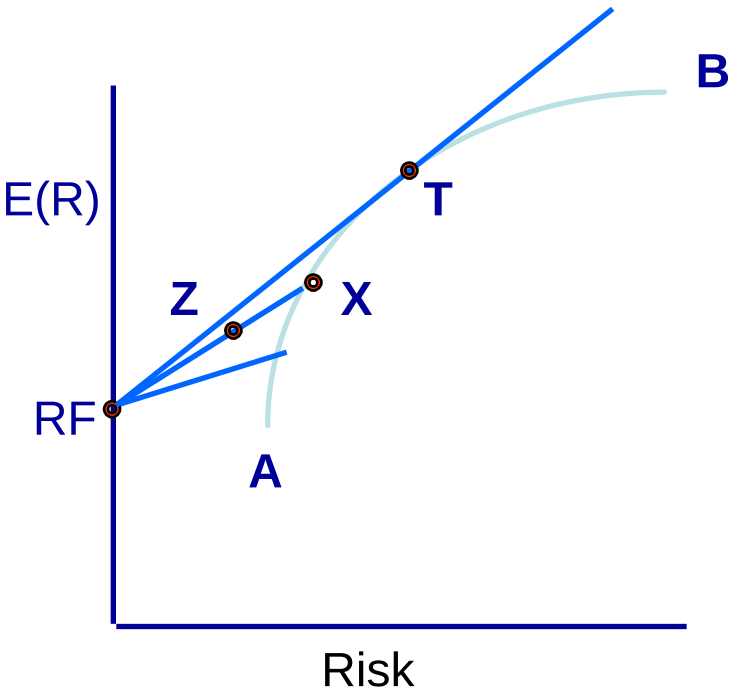
Interest paid on borrowed money (short-selling)

- Higher returns sought to cover expense
- Assume borrowing at RF

Risk will increase as the amount of borrowing increases

- Financial leverage

When riskless lending and borrowing is allowed, the efficient frontier is a straight line, tangent to the concave, risky-assets, efficient frontier of risky assets.

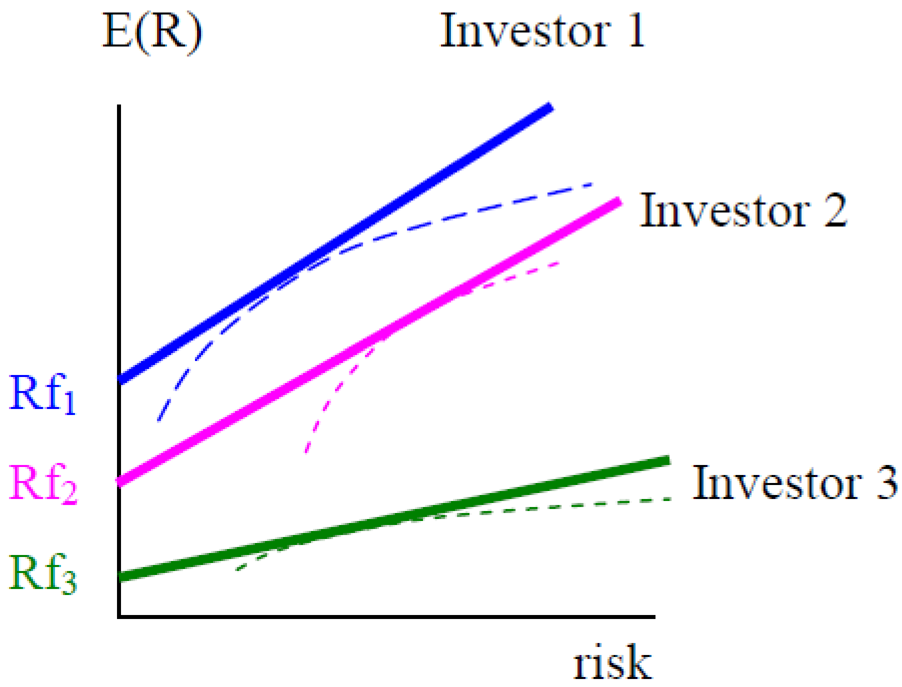



Portfolio T will be the only portfolio of risky assets held by investors facing this efficient frontier.

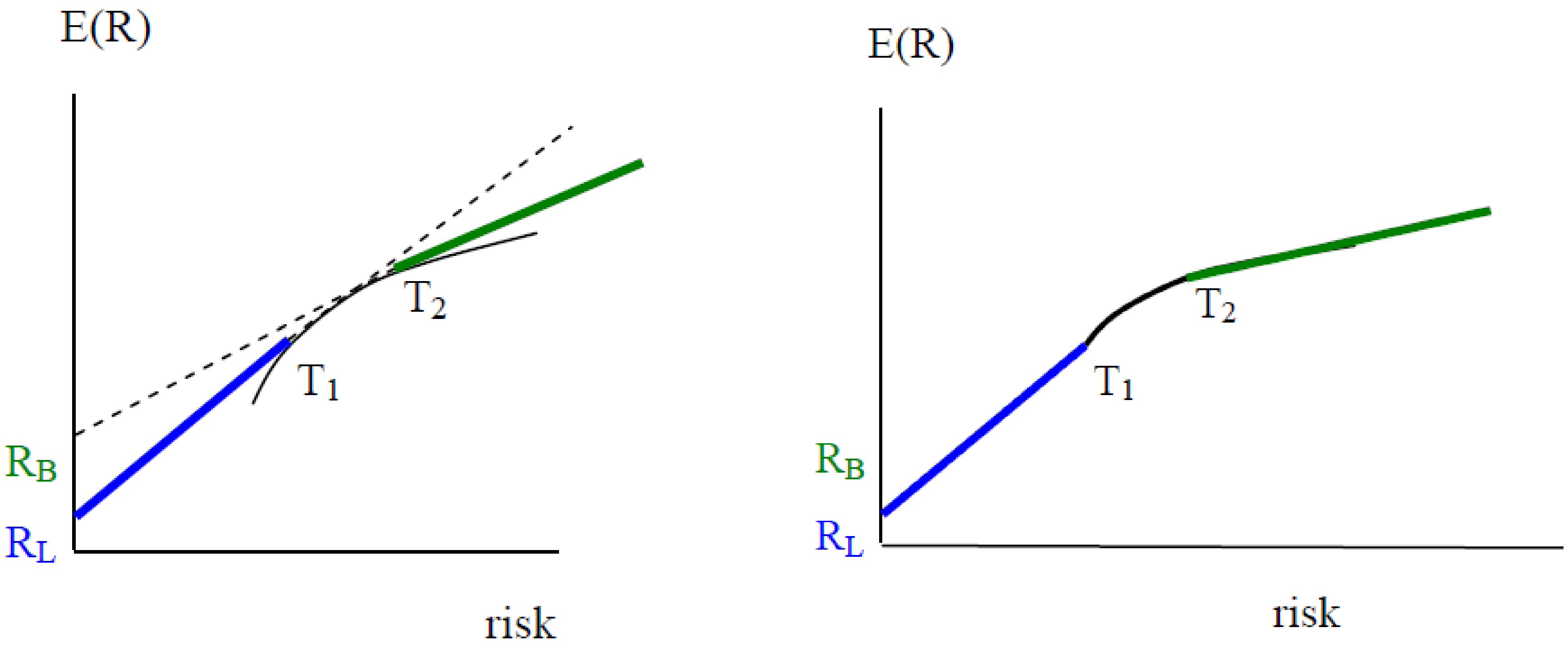
More risk averse investors (involved in risk-free lending) would select a portfolio along the segment RF-T.

Less risk averse investors will borrow funds and place their original capital plus the borrowed funds in risky portfolio T.

## Efficient frontiers for different investors and at different risk free rates



# Efficient frontiers at different borrowing and lending rates




RL to T1 = lending
From the right of T2 = borrowing
T1 and T2 = invest just in risky assets